The University of British Columbia

Final Examinations – Dec 13, 2011

Mathematics 221, Section 103

Instructor: K. Behrend

Time: 2.5 hours

Special instructions:

1. No books or notes or electronic aids allowed.
2. Show enough of your work to justify your answer. Show ALL steps.

Problem 1: Consider the system of equations in the variables x_1, x_2, x_3:

$$\begin{array}{llll} x_1 & +2x_2 & +3x_3 & = 1 \\ x_1 & +x_2 & +2x_3 & = 2 \\ 2x_1 & +3x_2 & +(5+t)x_3 & = 2 \end{array}$$

a) Determine all the values of t for which the system is consistent.

b) For those t for which the system is consistent, give the solution set in parametric form.

Problem 2: a) Determine the values of t for which the following matrix

$$A = \begin{pmatrix} 3 & 2 & t+3 \\ 3 & 4 & t+1 \\ 2 & 2 & t+1 \end{pmatrix}$$

is invertible.

b) Compute A^{-1} when $t = 0$.

Problem 3: a) Consider the matrix

$$A = \begin{pmatrix} -6 & 8 \\ -4 & 6 \end{pmatrix}$$

Compute the eigenvalues of A and a non-zero eigenvector for each eigenvalue.

b) With A as above, compute $\det(B)$, where $B = A^2 + 3A + 2I$.

Problem 4: Let

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 2 & 3 \end{pmatrix}$$

Then is A diagonalizable? Explain your answer.

b) True or false (explain your answer): If v is an eigenvector for the invertible matrix A, then v is also an eigenvector for the matrix A^{-1}.

Problem 5: a) Find the standard matrix of the linear transformation of $\mathbf{R}^3$ which reflects across the yz-plane.

b) Let

$$b_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad b_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad b_3 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Let T denote a linear transformation of $\mathbf{R}^2$ such that

$$T(b_1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad T(b_2) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Find $T(b_3)$, and give the matrix of T with respect to the standard basis of $\mathbf{R}^2$.

Problem 6: Consider the vectors

$$v_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \ v_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \ v_3 = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \ v_4 = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

a) Check that v_1, v_2, v_3, v_4 is an orthogonal basis of $\mathbf{R}^4$.

b) Let $b = (1, 2, 3, 4)^T$. Then write b as a linear combination of the vectors v_1, v_2, v_3, v_4.

Problem 7: Let

$$A = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{6} \end{pmatrix} \qquad \text{and let} \qquad x_0 = \begin{pmatrix} 5 \\ 5 \\ 6 \end{pmatrix}$$

Let $x_n = A^n x_n$. Then find $x_{100} = A^{100} x_0$. What happens to x_n as n becomes very large?

Problem 8: Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & -1 & 2 & 4 \\ 0 & 1 & -2 & 0 & -1 \\ -1 & -2 & 5 & 1 & 4 \\ 1 & 0 & -1 & 1 & 2 \end{pmatrix}$$

a) Find a basis for the column space of A.

b) Find the dimension of the nullspace of A.