# Final Exam

December 18, 2007, 15:30–18:00

**No books. No notes. No calculators. No electronic devices of any kind.**

**Name (block letters)** ______________________________

**Student Number** ______________________________

**Signature** ______________________________

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | total/64 |
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This exam has 10 problems. The first 9 problems are common to all three sections, the last problem is section-specific.

**Problem 1.** (6 points)
Find a basis for the null space of the matrix

$$A = \begin{pmatrix} 2 & -4 & 1 & t \\ 1 & -2 & 2 & t \\ 1 & -2 & 1 & 2t \\ 1 & -2 & 1 & t \end{pmatrix}$$

Your answer will depend on the value of $t$.

**Problem 2.** (4 points)
Find the inverse of the matrix

$$B = \begin{pmatrix} 2 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

**Problem 3.** (6 points)
Find a $2 \times 2$ matrix $A$ such that

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} A \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

**Problem 4.** (4 points)

Decide whether or not $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is contained in $W = \text{Span}\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\}$.

**Problem 5.** (6 points)

(a) (4 points) Find the determinant of the matrix

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 7 & 10 \\ 1 & 4 & 10 & t \end{pmatrix}$$

(b) (2 points) For what values of $t$ is $B$ invertible?

**Problem 6.** (6 points)

(a) Explain why $\mathcal{B} = \{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\}$ forms a basis for $\mathbb{R}^2$.

(b) Find the coordinate vector of $\begin{pmatrix} 7 \\ 10 \end{pmatrix}$ in the basis $\mathcal{B}$.

(c) Suppose the standard matrix of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is

$$\begin{pmatrix} 2 & -3 \\ 0 & 2 \end{pmatrix}$$

Find the matrix of $T$ with respect to the basis $\mathcal{B}$, i.e., find $[T]_{\mathcal{B}}$.

**Problem 7.** (8 points)
Consider the matrix

$$A = \begin{pmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{pmatrix}.$$

(a) (2 points) Verify that 1 is an eigenvalue of $A$.
(b) (3 points) Find all eigenvalues of $A$.
(c) (3 points) For each eigenvalue, find the dimension of the corresponding eigenspace.

**Problem 8.** (8 points)
For each of the linear maps $T : \mathbb{R}^2 \to \mathbb{R}^2$ do the following: find a basis of $\mathbb{R}^2$ consisting of eigenvectors of $T$, or explain why this is not possible.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by reflection across the line $y = \frac{1}{2}x$.
(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by rotation clockwise 5 degrees.
(c) the *dilation* $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\vec{v}) = 5\vec{v}$, for all $\vec{v} \in \mathbb{R}^2$.
(d) the *shear* in the $y$-direction whose matrix is

$$\begin{pmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{pmatrix}$$

**Problem 9.** (8 points)
On a remote planet, moisture is present in clouds, on the continents, and in the seas. Each year 80% of the cloud moisture falls onto the land and 10% falls into the seas. Each year 15% of the land moisture evaporates directly into the clouds and 65% runs into the seas. Each year 70% of the sea moisture evaporates into the clouds.

Assume we know also that the total amount of water or moisture on the planet is 86 trillion litres.

(a) Draw a diagram of the moisture transfer on this planet. (1 point)

(b) Write a system of linear equations describing the moisture transfer using the variables $c$, $l$, and $s$ for water content in clouds, on land, and in the seas, respectively. Find the transition matrix of this dynamical system. (2 points)

(c) Suppose the planets moisture distribution is in equilibrium. What is the annual precipitation (volume of water falling onto land from the clouds)? (3 points)

(d) A large meteor falls onto the planet, causing all sea water to evaporate into the clouds. (The impact has no other effect on the moisture distribution, but it does cause mass extinction of aquatic species.) Write down the matrix which represents the change in the state vector $\left(\begin{smallmatrix} c \\ l \\ s \end{smallmatrix}\right)$ caused by the impact. (2 points)

**Problem 10.** (8 points)
A discrete dynamical system is defined by

$$\begin{aligned} x_{n+1} &= x_n + 3y_n \qquad & x_0 &= 5 \\ y_{n+1} &= 2x_n + 2y_n \qquad & y_0 &= 10 \end{aligned}$$

(a) (6 points) Find explicit formulas for $x_n$ and $y_n$, with the given initial condition.

(b) (1 point) Find the limiting growth rate of $x$, i.e., find $\lim_{n\to\infty} \frac{x_{n+1}}{x_n}$.

(c) (1 point) Find the limit $\lim_{n\to\infty} \frac{x_n}{y_n}$.