The University of British Columbia
Final Examination - April 20, 2007

Mathematics 221

Sections 201, 202, 203
Instructors: Dr. Macasieb, Dr. Tsai, and Dr. Liu

Closed book examination — Time: 2.5 hours

Name ______________________ **Signature** ______________________

Student Number ______________

Special Instructions:
- Be sure that this examination has 12 pages. Write your name on top of each page.
- No calculators or notes are permitted.
- Show all your work. Unsupported solutions deserve no mark.
- In case of an exam disruption such as a fire alarm, leave the exam papers in the room and exit quickly and quietly to a pre-designated location.

Rules governing examinations

- Each candidate should be prepared to produce her/his library/AMS card upon request.
- No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of examination.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Making use of any books, papers, or memoranda, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates.
- Smoking is not permitted during examinations.

1		12
2		10
3		10
4		12
5		10
6		12
7		7
8		12
9		15
Total		100

1. [12pt] Consider the following linear system

$$\begin{aligned} x + 3y - 2z + 2w &= 1 \\ y + z - 2w &= 2 \\ x + 2y - 2z + aw &= 0 \\ 2x + 8y - z + w &= b \end{aligned}$$

For which values of a and b, if any, does the system have: (Justify you answers!!)
(i) No solution? (ii) Exactly one solution?
(iii) Exactly two solutions? (iv) More than two solutions?

2. [10pt] Let S be the map in $\mathbf{R}^3$ which rotates points about the x_1-axis by an angle $\pi/2$ (the axes are oriented by the right hand rule). Let T be the map in $\mathbf{R}^3$ which translates points by the formula $T(x_1, x_2, x_3)^T = (x_1 + 1, x_2 - 1, x_3)^T$. One of them is a linear transformation and the other is not.
(i) Decide and justify which one is NOT a linear transformation.
(ii) You may assume the other one is a linear transformation. Find its standard matrix.

3. [10pt] For what values of k is the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & k \end{bmatrix}$ invertible? When it is invertible, find its inverse.

4. [12pt] Let $W = \left\{ \begin{bmatrix} b+2c-d \\ 2b+4c-d \\ d \\ -b-2c+d \end{bmatrix} \;\middle|\; b, c, d \text{ real} \right\}$.

(i) Find a matrix A such that Col $A = W$.

(ii) Find a basis for W.

(iii) If $B = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & k \\ 1 & 1 & 1 & 3 \end{bmatrix}$ and dim (Row B) = 2, find the value of the constant k.

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5. [10pt] Let $A = \begin{bmatrix} x & 1 & 1 & 1 & 1 \\ 1 & x & 1 & 1 & 1 \\ 1 & 1 & x & 1 & 1 \\ 1 & 1 & 1 & x & 1 \\ 1 & 1 & 1 & 1 & x \end{bmatrix}$. Find all values of x such that A is not invertible.

6. [12pt] Let $\mathbb{P}_2$ be the vector space of polynomials of degree at most 2.

(i) The set $\mathcal{B} = \{1+t, 1+t^2, t+t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector $[2+t-t^2]_{\mathcal{B}}$.

(ii) The set $\mathcal{C} = \{1+t^2, t+t^2, 1+t\}$ is also a basis for $\mathbb{P}_2$. Find $\vec{p}(t)$ in $\mathbb{P}_2$ such that $\vec{p}(1) = 1$ and $[\vec{p}(t)]_{\mathcal{B}} = [\vec{p}(t)]_{\mathcal{C}}$.

7. [7pt] Suppose a 2×2 matrix A has eigenvalues 1 and 1/2 with corresponding eigenvectors

$$\vec{v_1} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \text{ and } \vec{v_2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

What is $\lim_{k \to \infty} A^k$?

8. [12pt] Suppose

$$\vec{w_1} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \vec{w_2} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \vec{w_3} = \begin{bmatrix} -1 \\ -1 \\ 7 \end{bmatrix}, \vec{y} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

Let $W = \text{Span}\{\vec{w_1}, \vec{w_2}, \vec{w_3}\}$.
(i) Determine the dimension of W and find a basis for W.
(ii) Find an orthogonal basis for W, and the orthogonal projection of $\vec{y}$ onto W.
(iii) What is the shortest distance from $\vec{y}$ to W?

9. [8/2/5pt] The matrix $M = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$.

(i) Verify that M has eigenvalues 0 and 3, and find the corresponding eigenspaces.
(ii) What is the rank of M?
(iii) Is M diagonalizable? Is there an orthogonal set of eigenvectors of M that forms a basis of $\mathbb{R}^3$? Justify your answers.

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