Be sure this examination has 2 pages.

The University of British Columbia
Final Examinations - December 2005

Mathematics 221: *Matrix Algebra*

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Closed book examination. **Time: 2.5 hours = 150 minutes.**

Special Instructions: **No aids allowed.** Write your answers in the answer booklet(s). If you use more than one booklet, put your name and the number of booklets used on each booklet. **Show enough of your work to justify your answers.**

1. (15 points) Consider the system of equations

$$\begin{array}{ccccccc} x & + & y & + & 2z & = & 0 \\ x & + & 2y & + & z & = & q \\ 2x & + & (2+q)y & + & 3z & = & 1, \end{array}$$

where the constant q is *not* specified. For what values of q does this system have:
(i) No solution?
(ii) Exactly one solution?
(iii) Exactly two solutions?
(iv) More than two solutions?
Remember to provide some calculations and/or other reasons to support your answers.

2. (10 points) Consider the following transformations on R^2. First, rotate each input vector by $+90°$. Then reflect the result of the first step in the x_1-axis. Show that combining these two steps has the same effect as simply reflecting each input vector in a suitable line, and find that line.

3. (15 points) Let W be the subspace of R^3 spanned by the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 4 \\ 1 \\ 4 \end{bmatrix}, \qquad \vec{v}_3 = \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix}.$$

Do the following three things in some order, and then answer the fourth part.
(a) Find a basis for W and find the dimension of W.
(b) Determine whether the vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ belongs to W.
(c) Find a basis for $W^{\perp}$, the subspace of all vectors in R^3 that are orthogonal to all vectors in W.
(d) Does combining your basis in part (a) and the basis in part (c) give a basis for R^3? Explain briefly.

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4. (15 points) On Mars, there are three political parties, named X, Y and Z. Between any two elections, votes move between parties in the following way. Suppose the number of votes for party X in the k-th election is x_k, and that y_k and z_k are the numbers of votes for parties Y and Z in that election. Then in the the next election, the numbers of votes are given by the equation

$$\vec{v}_{k+1} = \begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 3x_k + 2y_k + z_k \\ 2x_k + 2y_k + 2z_k \\ x_k + 2y_k + 3z_k \end{bmatrix}.$$

(a) Find $\vec{v}_5$ given that $\vec{v}_3 = \begin{bmatrix} 54 \\ 36 \\ 18 \end{bmatrix}$ in Martian units.

(b) Is there any nontrivial arrangement of votes that would stay the same from one election to the next? If so, find such an arrangement. If not, say why there is no such arrangement.

(c) Suppose that $\vec{v}_2$ is given, but $\vec{v}_1$ is *not* given. Is there only one possible choice of $\vec{v}_1$ that leads to the next result $\vec{v}_2$? Why or why not?

5. (20 points) Are the following statements always true or sometimes false? Give reasons for your answers

(a) The vectors running from the origin to the plane with equation $x + 2y + 3z = 4$ form a subspace of R^3.

(b) If a matrix A with 5 rows and 9 columns has rank 3, then each equation $A\vec{x} = \vec{b}$, where $\vec{b}$ is in R^5, has a solution with 6 free variables.

(c) If $\det A = 0$, then $\det(A^2 + 5A) = 0$ too.

(d) If B is a matrix with real entries, and if λ is an eigenvalue of B^2, then $\lambda \geq 0$.

6. (15 points) Consider the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

(a) Verify that $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector of A.

(b) Verify that 0 is an eigenvalue of A, by finding a corresponding eigenvector.

(c) Find an orthogonal set of eigenvectors of A that form a basis for R^3.

7. (10 points) Suppose that a 2-by-2 matrix C has eigenvectors $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$ with eigenvalues 3 and -1 respectively. Let $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and find $C^{99}(C - 3I)\vec{v}$.

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