*This final exam has **8 questions** on **12 pages**, for a total of 100 marks.*

Duration: 2 hours 30 minutes

Full Name (including all middle names): ______________________________

Student-No: ______________________________

Signature: ______________________________

UBC Rules governing examinations:

1. Each candidate should be prepared to produce his/her library/AMS card upon request.
2. No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour or the *last 15 minutes* of the examination. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in the examination questions.
3. Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination, and shall be liable to disciplinary action:

   a) Making use of any books, papers or memoranda, other than those authorised by the examiners.

   b) Speaking or communicating with other candidates.

   c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness will not be received.
4. Smoking is not permitted during examinations.

Question:	1	2	3	4	5	6	7	8	Total
Points:	30	8	10	10	10	10	10	12	100
Score:									

Please read the following points carefully before starting to write.

- Read all the questions carefully before starting to work.
- You should give complete arguments and explanations for all your answers and calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)
- You may not leave during the first 30 minutes or final 15 minutes of the exam.

30 marks 1. Determine whether the following statements are true or false — Put True or False in the boxes. Justify your answers (Just writing "True" or "False" is not sufficient).

(a) $\forall n \in \mathbb{N}$, $n^2 + 3n + 8$ is even.

(b) There are only finitely many denumerable subsets of $\mathbb{N}$.

(c) If the sequence $a_n \to +\infty$ as $n \to \infty$, then the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges.

(d) For every two sets A and B, if $A \subset B$ (proper subset), then $|A| < |B|$.

(e) For every two irrational numbers a and b, we have that a^b is also irrational.

(f) For every two disjoint sets A and B, we have that $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.
($\mathcal{P}$ stands for power set)

(g) Let A and B be nonempty sets and let $f : A \to B$ be a function. If $D \subseteq B$, then $f(f^{-1}(D)) = D$.

(h) Given a nonempty set A and a function $f : A \to A$, if f is an equivalence relation on A then it is the identity function on A .

(i) Let $0 \leq a_n \leq b_n$ for all natural numbers n. If the sequence $\{b_n\}$ converges, then the sequence $\{a_n\}$ converges.

(j) A sequence $\{a_n\}$ is said to diverge if $\forall L \in \mathbb{R}$, $\exists \epsilon > 0$ s.t. $\exists N \in \mathbb{N}$, $\exists n > N$, $|a_n - L| \geq \epsilon$.

8 marks 2. (a) Let $f : A \to B$ be a surjection and let $D_1, D_2 \subseteq B$. Show that if $f^{-1}(D_1) \subseteq f^{-1}(D_2)$ then $D_1 \subseteq D_2$.

(b) Construct an example that shows the above is not true when f is not a surjection.

$\boxed{\text{10 marks}}$ 3. Let $x \in [3, +\infty)$, use induction to prove that for all natural numbers n, we have

$$\frac{(n+2)(n+1)n}{6}\, x^3 \leq (1+x)^{n+2}$$

10 marks 4. A relation R is defined on $\mathbf{Z}$ by $a \, R \, b$ if $7a^2 \equiv 2b^2 \pmod 5$. Prove that R is an equivalence relation. Determine the distinct equivalence classes $[0]$ and $[1]$, simplify your answer as much as possible.

$\boxed{\text{10 marks}}$ 5. Let $f : \mathbf{R} \to \mathbf{R}$ be a function which is defined by $f(x) = \dfrac{2x}{1+x^2}$.
Show that $f(\mathbf{R}) = [-1, 1]$.

Hint — prove that $f(\mathbf{R}) \subseteq [-1, 1]$ and $[-1, 1] \subseteq f(\mathbf{R})$.

$\boxed{\text{10 marks}}$ 6. Assume that A, B and C are three nonempty sets such that $A \cap C = \emptyset$ and $B \cap C = \emptyset$. Prove that if $|A| = |B|$ then $|A \cup C| = |B \cup C|$.

10 marks 7. Prove that $\dfrac{n+2}{2n^2+3n} \to 0$ as $n \to \infty$, by using the definition of the limit for a sequence.

Hint — you have to use the $(\epsilon - N)$ definition of the limit in your proof, no marks will be given if you use limit laws.

$\boxed{\text{12 marks}}$ 8. (a) Mark each sequence Convergent or Divergent in the box. Calculate the limit if convergent (you can use the limit laws). Justify your answer.

(i) $\left\{ \dfrac{2n^2 - n + 1}{5n^2 + 1} \right\}_{n \in \mathbb{N}}$

$\boxed{\phantom{\text{Convergent}}}$

(ii) $\left\{ \sqrt{n^2 + n} - n \right\}_{n \in \mathbb{N}}$

$\boxed{\phantom{\text{Convergent}}}$

(b) Mark each series Convergent or Divergent. Calculate the sum if convergent. Justify your answer.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{(n+1)^2}$

$\boxed{\phantom{\text{Convergent}}}$

(ii) $\displaystyle\sum_{n=2}^{\infty} \left(\frac{1}{n(n+1)} - \left(\frac{1}{2}\right)^n \right)$ Hint —be careful, this sum starts at $n = 2$.

$\boxed{\phantom{\text{Convergent}}}$

This page has been left blank for your workings and solutions.