

The University of British Columbia

Final Examination - December 12, 2012

Mathematics 220

All Sections

Closed book examination

Time: 2.5 hours

Last Name: _____, First: _____ Signature _____

Student Number _____

Special Instructions:

No books, notes or calculators are allowed.

Student Conduct during Examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (a) speaking or communicating with other candidates, unless otherwise authorized;
 - (b) purposely exposing written papers to the view of other candidates or imaging devices;
 - (c) purposely viewing the written papers of other candidates;
 - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total		100

[10] 1. For all natural numbers n , let

$$S_n = \{1, n, n^2, n^3, \dots\}.$$

Then the sets $A = \bigcup_{n \in \mathbb{N}} S_n$ and $B = \bigcap_{n \in \mathbb{N}} S_n$ are familiar to you. Describe A and B in as simple terms as possible, and justify your answers.

[10] **2.** Prove that for every two real numbers a, b such that $a \neq b$, either $\frac{a+b}{2} > a$ or $\frac{a+b}{2} > b$.

[10] **3.**

- (i) Disprove the statement: for real numbers a and b , if a is irrational and b is irrational, then $a - b$ is irrational.
- (ii) Let x be a positive real number. Prove that if $x - \frac{3}{x} > 2$, then $x > 3$.

[10] 4. Define a relation R on \mathbb{Z} by:

$$xRy \text{ if } xy \equiv 0 \pmod{4}.$$

- (i) Is R reflexive? Why or why not?
- (ii) Is R symmetric? Why or why not?
- (iii) Is R transitive? Why or why not?

[10] 5. Let P , Q and R be statements.

(i) Show that $((P \wedge Q) \Rightarrow R) \equiv ((P \wedge (\sim R)) \Rightarrow (\sim Q))$.

(ii) If the implication $((\sim P) \vee Q) \Rightarrow (\sim R)$ is false and P is true, determine the truth values of Q and R .

[10] **6.** Let $A = \{n \in \mathbb{Z} \mid n^2 - 2n + 9 \equiv 0 \pmod{8}\}$ and $B = \{n \in \mathbb{Z} \mid n \equiv 1 \pmod{4}\}$. Prove that $B \subseteq A$.

[10] 7. Prove that

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$$

whenever n is a positive integer greater than 1.

[10] **8.** Prove that the function $f : \mathbb{R} - \{1\} \rightarrow \mathbb{R} - \{2\}$ defined by $f(x) = \frac{2x + 3}{x - 1}$ is bijective.

[10] **9.** Give an $\epsilon - \delta$ proof that $\lim_{x \rightarrow 2} (2x^2 - 3x + 1) = 3$.

[10] **10.** Prove that $|(0, 1)| = |\mathbb{N} \cup (0, 1)|$, where $(0, 1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$.