# The University of British Columbia

Final Examination - December 12, 2012

## Mathematics 220

All Sections

Closed book examination                                    Time: 2.5 hours

**Last Name:** _____ , **First:** _____ **Signature** _____

**Student Number** _____

**Special Instructions:**

No books, notes or calculators are allowed.

| | | |
|---|---|---|
| 1 | | 10 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| 10 | | 10 |
| Total | | 100 |

[10] **1.** For all natural numbers $n$, let

$$S_n = \{1, n, n^2, n^3, \ldots\}.$$

Then the sets $A = \bigcup_{n \in \mathbb{N}} S_n$ and $B = \bigcap_{n \in \mathbb{N}} S_n$ are familiar to you. Describe $A$ and $B$ in as simple terms as possible, and justify your answers.

[10] **2**. Prove that for every two real numbers $a$, $b$ such that $a \neq b$, either $\dfrac{a+b}{2} > a$ or $\dfrac{a+b}{2} > b$.

[10] **3.**

(i) Disprove the statement: for real numbers $a$ and $b$, if $a$ is irrational and $b$ is irrational, then $a - b$ is irrational.

(ii) Let $x$ be a positive real number. Prove that if $x - \dfrac{3}{x} > 2$, then $x > 3$.

[10] **4.** Define a relation $R$ on $\mathbb{Z}$ by:

$$xRy \text{ if } xy \equiv 0 \pmod 4.$$

(i) Is $R$ reflexive? Why or why not?

(ii) Is $R$ symmetric? Why or why not?

(iii) Is $R$ transitive? Why or why not?

[10] **5.** Let $P$, $Q$ and $R$ be statements.

(i) Show that $\big((P \wedge Q) \Rightarrow R\big) \equiv \Big(\big(P \wedge (\sim R)\big) \Rightarrow (\sim Q)\Big)$.

(ii) If the implication $\big((\sim P) \vee Q\big) \Rightarrow (\sim R)$ is false and $P$ is true, determine the truth values of $Q$ and $R$.

[10] **6**. Let $A = \left\{ n \in \mathbb{Z} \mid n^2 - 2n + 9 \equiv 0 \pmod{8} \right\}$ and $B = \left\{ n \in \mathbb{Z} \mid n \equiv 1 \pmod{4} \right\}$. Prove that $B \subseteq A$.

[10] **7.** Prove that

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$$

whenever $n$ is a positive integer greater than 1.

[10] **8.** Prove that the function $f : \mathbb{R} - \{1\} \to \mathbb{R} - \{2\}$ defined by $f(x) = \dfrac{2x + 3}{x - 1}$ is bijective.

[10] **9**. Give an $\epsilon - \delta$ proof that $\lim_{x \to 2}(2x^2 - 3x + 1) = 3$.

[10] **10**. Prove that $\left|(0,1)\right| = \left|\mathbb{N} \bigcup (0,1)\right|$, where $(0,1) = \left\{x \in \mathbb{R} \mid 0 < x < 1\right\}$.