## The University of British Columbia

Math 220 — Introduction to Mathematical Proof

## 2011, April 21

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First Name:	Last Name:					
Student id.		Section (circle):	201	202		

## Instructions

- This exam consists of **10 questions** worth a total of 100 points.
- Justify all answers, show all work and calculations. Extra paper is available as needed.
- No calculators or other aids are permitted.
- Make sure this exam has 12 pages excluding this cover page.
- Duration: 2 hours 30 minutes.

Good luck, and enjoy your summer.

1. Each candidate should be prepared to produce his library/AMS card upon request.

## 2. Read and observe the following rules:

No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.

Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

(a) Making use of any books, papers or memoranda, other than those authorized by the examiners.

(b) Speaking or communicating with other candidates.

(c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

3. Smoking is not permitted during examinations.

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	10	8	10	5	5	5	15	12	15	15	100
Score:											

(10 marks) 1. (a) Let  $f : A \to B$  be a function and let  $C \subseteq A$  and  $D \subseteq B$ . Define the sets  $f^{-1}(D)$  and f(C).

(b) Define the supremum of a set S of real numbers.

(c) State the converse of the statement

"If I will win the lottery then I will buy a new car."

(d) Write the negation of the statement

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{N} \text{ s.t. } x^2 + y < x + y^2.$$

(e) Define what it means for a function  $f: A \to B$  to be injective.

(f) Define what it means for the sequence  $\{a_n\}$  to converge to a number L.

(g) Define what it means for the series  $\sum_{n=1}^{\infty} a_n$  to converge.

(h) Define what it means for a set S to be countable.

(i) State the principle of mathematical induction.

(j) Let A be the set  $\{1, x, y\}$ . What is the power set  $\mathcal{P}(A)$ ?

(8 marks) 2. For each of the following subsets of ℝ write its supremum and infimum if they exist. If they do not exist write "None". You do not need to prove your answers.

(a) 
$$\left\{x \in \mathbb{R} \text{ s.t. } -1 < x < 5\right\}$$
 (b)  $\left\{x \in \mathbb{Q} \text{ s.t. } 3 \le x^2 \le 7\right\}$   
(c)  $\bigcap_{n=1}^{\infty} \left[2 + \frac{1}{n}, 6 - \frac{2}{n}\right]$  (d)  $\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, n\right]$ 

(10 marks) 3. Using the definition of convergence for sequences, prove that

(a) 
$$\lim_{n \to \infty} \frac{n^2 + 3n + 1}{n^2} = 1.$$
 (b)  $\lim_{n \to \infty} \frac{1 - 2\cos(n)}{n} = 0.$ 

(5 marks) 4. Prove that

 $P \implies (Q \lor R) \equiv (P \implies Q) \lor (P \implies R).$ 

(5 marks) 5. Prove that the function  $f : \mathbb{R} - \{1\} \to \mathbb{R} - \{2\}$  given by  $f(x) = \frac{2x}{x-1}$  is bijective.

(5 marks) 6. Let  $n \in \mathbb{Z}$ . Prove that  $n \equiv 3 \pmod{5}$  if and only if 3n + 1 is divisible by 5.

(15 marks) 7. (a) Let  $\{a_n\}$  be a sequence of real numbers defined by

$$a_1 = 1$$
 and  $a_{n+1} = 2a_n + 1$  for each  $n \in \mathbb{N}$ .

Prove that  $a_n = 2^n - 1$  for all  $n \in \mathbb{N}$ .

(b) Let  $\{b_n\}$  be a sequence defined by

$$b_1 = 2$$
 and  $b_{n+1} = \frac{b_n + \sqrt{b_n}}{2}$  for each  $n \in \mathbb{N}$ .

Prove that  $1 \leq b_n \leq 2$  for each  $n \in \mathbb{N}$ .

(c) Prove the sequence  $\{b_n\}$  above satisfies  $b_{n+1} \leq b_n$  for each  $n \in \mathbb{N}$ .

- (12 marks) 8. Let  $P \subset \mathbb{N}$  be the set of prime numbers  $P = \{2, 3, 5, 7, ...\}$ . Determine whether the following statements are true or false. Prove your answers ("true" or "false" is not sufficient).
  - (a)  $\forall m \in P, \forall n \in P, m + n \in P$ .
  - (b)  $\forall m \in P, \exists n \in P \text{ s.t. } m + n \in P.$
  - (c)  $\exists m \in P \text{ s.t. } \forall n \in P, m + n \in P.$
  - (d)  $\exists m \in P \text{ s.t. } \exists n \in P \text{ s.t. } m + n \in P.$

(15 marks) 9. Let  $f: A \to B$  and  $g: B \to A$  be functions so that  $g \circ f$  is a surjective function.

- (a) Prove that g is surjective.
- (b) Give an example of sets A, B and functions f, g as above such that f is not surjective.
- (c) Prove or disprove that  $f \circ g$  must be surjective.

- (15 marks) 10. (a) Let  $A \subset \mathbb{R}$  be a bounded set of real numbers, and let  $B \subseteq A$  be a nonempty subset of A. Prove that B is also bounded and that  $\inf(A) \leq \sup(B) \leq \sup(A)$ .
  - (b) Let  $\{a_n\}_{n=1}^{\infty}$  be a bounded sequence of real numbers. For each  $n \in \mathbb{N}$  define

$$b_n = \sup\left(\{a_m : m \in \mathbb{N} \text{ s.t. } m \ge n\}\right)$$

Prove that  $\{b_n\}_{n=1}^{\infty}$  is a convergent sequence. (You may use part (a)).

Question: \_\_\_\_\_