This final exam has ***10 questions*** *on* ***15 pages****, for a total of 80 marks.*

Duration: 2 hours 30 minutes

Full Name (including all middle names): ______________________________

Student-No: ______________________________

Signature: ______________________________

UBC Rules governing examinations:

1. Each candidate should be prepared to produce his/her library/AMS card upon request.

2. No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour or the *last 15 minutes* of the examination. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in the examination questions.

3. Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination, and shall be liable to disciplinary action:

   a) Making use of any books, papers or memoranda, other than those authorised by the examiners.

   b) Speaking or communicating with other candidates.

   c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness will not be received.

4. Smoking is not permitted during examinations.

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	8	9	8	8	9	8	9	7	6	8	80
Score:											

Please read the following points carefully before starting to write.

- Read all the questions carefully before starting to work.
- With the exception of Q1, you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)
- You may not leave during the first 30 minutes or final 15 minutes of the exam.

8 marks 1. (a) Let $f : A \to B$ be a function and let $C \subseteq A$ and $D \subseteq B$.
Define the sets $f^{-1}(D)$ and $f(C)$.

(b) Define what it means for $g : S \to T$ to be a surjective function.

(c) State the completeness axiom.

(d) Define the infimum of a set of real numbers.

(e) Define what it means for the sequence $\{c_n\}$ to diverge to $+\infty$.

(f) Write the negation of the following statement

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{Q} \text{ s.t. } (\sqrt{x} \in \mathbb{N}) \Rightarrow (y^2 > x).$$

9 marks 2. (a) Let $a \in \mathbb{Z}$. Prove that a is even if and only if a^3 is even.

(b) Let A, B, C be sets. Prove the following distributive law from first principles

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

8 marks

3. Let $\mathbb{I}$ be the set of irrational numbers. That is $\mathbb{I} = \mathbb{R} - \mathbb{Q}$.

   (a) Prove that if $x \in \mathbb{I}$ then $-x \in \mathbb{I}$.

   (b) Prove that if $x \in \mathbb{I}$ then $2x \in \mathbb{I}$.

   (c) Determine whether the following four statements are true or false — explain your answers ("true" or "false" is not sufficient).

   (i) $\forall x \in \mathbb{I}, \forall y \in \mathbb{I}, x + y \in \mathbb{I}$.

   (ii) $\forall x \in \mathbb{I}, \exists y \in \mathbb{I}$ s.t. $x + y \in \mathbb{I}$.

   (iii) $\exists x \in \mathbb{I}$ s.t. $\forall y \in \mathbb{I}, x + y \in \mathbb{I}$.

   (iv) $\exists x \in \mathbb{I}$ s.t. $\exists y \in \mathbb{I}$ s.t. $x + y \in \mathbb{I}$.

8 marks 4. Let $f : A \to B$ and $g : B \to A$ be functions so that $g \circ f$ is an injection.

(a) Prove that f must be injective.

(b) Show that g need not be injective.

(c) Prove or disprove that the composition $f \circ g$ must be injective.

9 marks 5. (a) Prove that the set $S = \{x \in \mathbb{R} \text{ s.t. } \sin(x) = 0\}$ is countable.

(b) Prove that the following function is bijective

$$f : \mathbb{R} - \{-2\} \to \mathbb{R} - \{1\} \text{ defined by } f(x) = \frac{x+1}{x+2}.$$

(c) Explain why $|\mathbb{R} - \{-2\}| = |\mathbb{R} - \{1\}|$.

This page has been left blank for your workings and solutions.

8 marks 6. Use mathematical induction to prove the following.

(a) Prove that

$$\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$$

is true for all $n \in \mathbb{N}$.

(b) Prove that $4^n \geq (n+1)^2$ for all non-negative integers n.

9 marks

7. (a) For each subset of $\mathbb{R}$ give its supremum, maximum, infimum and minimum if they exist. If they do not exist write "none". You do not need to justify your answers.

the sets	sup	max	inf	min
$\{x \in \mathbb{R} \text{ s.t. } 2 \leq x < \pi\}$				
$\{x \in \mathbb{Q} \text{ s.t. } x \geq 0 \text{ and } x^2 \leq 5\}$				
$\{x \in \mathbb{Z} \text{ s.t. } x^2 \geq 4\}$				
$\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, n^2+1\right]$				

(b) Let A and B be non-empty subsets of the interval $[0,1]$. Now let C be the set

$$C = \{ab \text{ s.t. } a \in A, b \in B\}.$$

(i) Prove that $\sup A$ and $\sup B$ exist.

(ii) Suppose that $\inf A = \inf B = 0$ and $\sup A = \sup B = 1$, prove that 0 is a lower bound for C and 1 is an upper bound for C.

(iii) Suppose that $\sup A = \sup B = 1$, prove that $\sup C = 1$.

This page has been left blank for your workings and solutions.

7 marks

8. (a) Let $\{a_n\}$ be a sequence of positive numbers. Prove that if $\{a_n\}$ diverges to infinity then $\{1/a_n\}$ converges to 0.

(b) Decide whether the following sequence converges or diverges. Prove your answer using the definition of convergence.

$$c_n = \frac{2n^3 - 5n + 2}{3n^3 + 8n + 3}$$

6 marks 9. (a) Let $a_n \to 1$. Prove that there exists N such that if $n > N$ then $|a_n + 1| < 3$.

(b) Prove that if $a_n \to 1$ then $a_n^2 \to 1$.

8 marks 10. (a) Let $r \in \mathbb{R}$. Prove that if $\lim_{n\to\infty} s_n = s$ then $\lim_{n\to\infty}(r - s_n) = r - s$.

(b) Let $k, \ell \in \mathbb{N}$. Prove, using the definition of convergence, that $\dfrac{1}{kn+\ell} \to 0$.

(c) Using (a) and (b) prove that

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}.$$

This page has been left blank for your workings and solutions.