*This final exam has **9 questions** on **16 pages**, for a total of 100 marks.*

Duration: 2 hours 30 minutes

Full Name (including all middle names): ______________________

Student-No: ______________________

Signature: ______________________

UBC Rules governing examinations:

1. Each candidate should be prepared to produce his/her library/AMS card upon request.
2. No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour or the *last 15 minutes* of the examination. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in the examination questions.
3. Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination, and shall be liable to disciplinary action:

   a) Making use of any books, papers or memoranda, other than those authorised by the examiners.

   b) Speaking or communicating with other candidates.

   c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness will not be received.
4. Smoking is not permitted during examinations.

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	15	8	8	12	15	10	10	12	10	100
Score:										

Please read the following points carefully before starting to write.

- Give complete arguments and explanations for all your calculations; answers without justifications will not be marked — except where specifically stated.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.).
- You may not leave during the first 30 minutes or final 15 minutes of the exam.
- Read all the questions carefully before starting to work.
- Continue on the back of the previous page if you run out of space.

15 marks

1. Please give precise answers to the following:
   (a) Let $f : A \to B$ be a function and let $C \subseteq A$ and $D \subseteq B$. Define the sets $f^{-1}(D)$ and $f(C)$.

   (b) Define what it means for a function to be injective.

   (c) Define what it means for a function to be surjective.

   (d) Define what it means for a set to be countable.

   (e) State De Morgan's laws (in the context of logic).

(f) State the principle of mathematical induction.

(g) Define the sets $X = \bigcap_{\alpha \in I} S_\alpha$ and $Y = \bigcup_{\alpha \in I} S_\alpha$.

(h) Define the supremum of a set $A \subseteq \mathbb{R}$.

(i) Define what it means for the sequence $\{c_n\}$ to converge.

(j) Define what it means for the sequence $\{c_n\}$ to diverge to infinity.

8 marks 2. (a) Write the negation of the following statement:

$\forall x \in \mathbb{Z}$, if $x > 0$ then $\exists y \in \mathbb{Q}$ s.t. $x^2 > y$.

(b) Let A, B, C be sets. Prove the following distributive law from first principles:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

8 marks 3. Let $f : A \to B$ be a function and let $C \subseteq A$.

(a) Give an example for which $f^{-1}(f(C)) \nsubseteq C$.

(b) Now let $E \subseteq A$. Prove that if f is injective then $f(C \cap E) = f(C) \cap f(E)$.

12 marks

4. (a) Prove that $f : [3, \infty) \to [5, \infty)$ defined by $f(x) = x^2 - 6x + 14$ is a bijection.

(b) Prove that if A is a denumerable subset of $\mathbb{Z}$ then there exists a set B such that B is a proper subset of A and $|A| = |B|$.

This page has been left blank for your workings and solutions.

15 marks 5. Use mathematical induction to prove the following statements:

(a) for all $n \in \mathbb{N}$, $1 \cdot 2^1 + 3 \cdot 2^2 + 5 \cdot 2^3 + \cdots + (2n-1) \cdot 2^n = 6 + 2^n(4n-6)$.

(b) for all non-negative integers n, $5^n + 2\,(11^n)$ is a multiple of 3.

(c) for all $n \in \mathbb{N}$, $\displaystyle\sum_{j=1}^{n} j^3 > \frac{1}{4}n^4$.

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10 marks

6. Let $\mathbb{I}$ be the set of irrational numbers. That is $\mathbb{I} = \mathbb{R} - \mathbb{Q}$.

   (a) Prove the following statements:

   (i) Let $x \in \mathbb{R}$. If $x \in \mathbb{I}$ then $1/x \in \mathbb{I}$.

   (ii) Let $n \in \mathbb{Z}$ and $x \in \mathbb{R}$. If $x \in \mathbb{I}$ then $n + x \in \mathbb{I}$.

   (b) Decide whether the following statements are true or false. Prove your answers.
   Hint: The results from (a) will be of great help.

   (i) $\forall x \in \mathbb{I}, \forall y \in \mathbb{I}, xy \in \mathbb{I}$.

   (ii) $\forall x \in \mathbb{I}, \exists y \in \mathbb{I}$ s.t. $xy \in \mathbb{I}$.

10 marks

7. Decide whether the following sequences converge or diverge. Prove your answers using the definition of convergence.

   (a) $a_n = \cos\left(\frac{n\pi}{3}\right)$

   (b) $b_n = \frac{3n^2 - 4n}{n^2 + 5n + 2}$

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12 marks

8. (a) Suppose that $s_n \to 0$. Prove that if t_n is a bounded sequence then $\lim_{n\to\infty} (s_n t_n) = 0$.

(b) Let s_n be a convergent sequence such that $s_n < q$ for all $n \in \mathbb{N}$.
Prove that $\lim_{n\to\infty} s_n \leq q$.

10 marks

9. (a) Find the supremum, maximum, infimum and minimum of the following subsets of $\mathbb{R}$ (if they exist). If they do not exist write "none". You do not need to justify your answers.

set	supremum	maximum	infimum	minimum
$\{x \in \mathbb{R} \text{ s.t. } 7 < x \leq 10\}$				
$\{x \in \mathbb{Q} \text{ s.t. } x^2 < 3\}$				
$\{x \in \mathbb{Z} \text{ s.t. } x^2 < 3\}$				
$\bigcap_{n=1}^{\infty} (1 - 1/n, 4 + 1/\sqrt{n})$				

(b) Let A be a non-empty proper subset of $(0, 3)$. Note $A \neq (0, 3)$. Define

$$B = \{-a \ \text{ s.t. } \ a \in A\}$$

(i) Prove that $\inf A$ and $\sup A$ exist using the Completeness Axiom.

(ii) Prove that $(-\inf A)$ is an upper bound for B.

(iii) Prove that $\sup B = (-\inf A)$.

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