*This final exam has **9 questions** on **16 pages**, for a total of 100 marks.*

Duration: 2 hours 30 minutes

Section Number (please circle): 201 / 202

Full Name (including all middle names): ______________________

Student-No: ______________________

Signature: ______________________

UBC Rules governing examinations:

1. Each candidate should be prepared to produce his/her library/AMS card upon request.
2. No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour or the *last 15 minutes* of the examination. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in the examination questions.
3. Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination, and shall be liable to disciplinary action:

   a) Making use of any books, papers or memoranda, other than those authorised by the examiners.

   b) Speaking or communicating with other candidates.

   c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness will not be received.
4. Smoking is not permitted during examinations.

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	10	10	10	10	15	15	10	10	10	100
Score:										

Please read the following points carefully before starting to write.

- Give complete arguments and explanations for all your calculations; answers without justifications will not be marked — except where specifically stated.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.).
- You may not leave during the first 30 minutes or final 15 minutes of the exam.
- Read all the questions carefully before starting to work.
- Continue on the back of the previous page if you run out of space.

10 marks

1. Please give precise mathematical definitions of the following:

   (a) surjective function

   (b) an uncountable set

   (c) the well-ordering principle of $\mathbb{N}$

   (d) the principle of mathematical induction

   (e) the sets $X = \bigcap_{\alpha \in I} S_\alpha$ and $Y = \bigcup_{\alpha \in I} S_\alpha$

(f) Define the infimum of a set of real numbers

(g) Let $f : A \to B$ be a function and let $D \subseteq B$. Define the set $f^{-1}(D)$.

(h) Define what it means for the sequence $\{c_n\}$ to converge

(i) Define what it means for the sequence $\{c_n\}$ to diverge to infinity

(j) Let $\{c_n\}$ be a sequence. Define what it means for $\displaystyle\sum_{n=1}^{\infty} c_n$ to converge

10 marks 2. (a) Let $\alpha, \beta \in \mathbb{R}$ and consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \alpha + \beta x$. For what values of α, β is this function bijective?

(b) Let $f : A \to B$ be an injective function and let $C_1, C_2 \subseteq A$.
Prove that $f(C_1 \cap C_2) = f(C_1) \cap f(C_2)$.

(c) Give an example to shows that the equation in (b) can fail when f is not injective.

10 marks 3. (a) Negate the following statement

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \text{ if } x < y \text{ then } \exists q \in \mathbb{Q} \text{ such that } x + q < y.$$

(b) Prove or disprove the following statement.

Let $a, b \in \mathbb{Z}$. If $a^2 + b^2$ is a perfect square then a and b are odd integers.

(c) Prove or disprove the following statement

If x is an irrational number then $x^{1/3}$ is also irrational.

(d) Let A, B, C, D be non-empty sets. Prove that

if $|A| \leq |C|$ and $|B| \leq |D|$ then $|A \times B| \leq |C \times D|$.

This page has been left blank for your workings and solutions.

10 marks 4. Let $f : (0, \infty) \to \mathbb{R}$ be a function defined by $f(x) = x - \frac{1}{x}$.

(a) Prove that f is bijective.

(b) Prove that $|(0, \infty)| = |\mathbb{R}|$.

15 marks

5. Prove the following results using induction.

   (a) $\displaystyle\sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}$ for all $n \in \mathbb{N}$.

   (b) $7^n - 2^n$ is divisible by 5 for all $n \in \mathbb{N}$.

   (c) $3^n > n^2$ for all $n \in \mathbb{N}$.
   Hint for (c)— Prove $n = 1$ first and then use induction to prove $n \geq 2$

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15 marks 6. (a) Simplify as much as possible the n^{th} partial sum of the series $\sum_{k=1}^{\infty} \log\left(1+\frac{1}{k}\right)$.

(b) Prove, from first principles, that the sequence $\left\{\frac{n^2+n-3}{n^2+4n+5}\right\}$ converges to 1

(c) Let $b, c, d \in \mathbb{R}$ with $c \neq 0$ and $d > 0$. Prove, from first principles, that the sequence $\left\{ b + \frac{c}{dn+1} \right\}$ converges to b.

(d) Prove that $\sum_{n=1}^{\infty} \frac{3}{16n^2 - 8n - 3}$ converges and find its limit.
Hint — part (c) may help you.

10 marks

7. Let $\{a_n\}$ be a convergent sequence with $a_n \to a$.

   (a) Prove that there is some $N \in \mathbb{N}$ so that if $n > N$ then $|a_n| \leq |a| + 1$.

   (b) Hence (or otherwise) prove that that $\lim \left(\frac{a_n}{n}\right) = 0$.

10 marks 8. Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences. Further, suppose $a_n \to a$ and $b_n \to b$.

(a) Prove that if $a \neq b$ then there exists some $N \in \mathbb{N}$ so that if $k > N$ then $a_k \neq b_k$.

(b) Now suppose that $a = b$. Prove or disprove that there is some $k \in \mathbb{N}$ so that $a_k = b_k$.

10 marks

9. (a) For each subset of $\mathbb{R}$ give its supremum, maximum, infimum and minimum if they exist. If they do not exist write "none". You do not need to justify your answers.

(i) $\{x \in \mathbb{R} \text{ such that } 6 \leq x < 12\}$ (ii) $\{x \in \mathbb{Q} \text{ such that } x^2 < 2\}$

(iii) $\displaystyle\bigcup_{n=1}^{\infty} (1 - 1/n, 3 + 1/n)$ (iv) $\displaystyle\bigcap_{n=1}^{\infty} (1 - 1/n, 3 + 1/n)$

(b) Let A be a non-empty subset of $[2, 5]$. Define

$$B = \left\{ \frac{1}{a} \text{ such that } a \in A \right\}$$

(i) Prove that $\inf A, \sup A, \inf B$ and $\sup B$ all exist.
(ii) Prove that $(\inf A)^{-1}$ is an upper bound for B.
(iii) Prove that $\sup B = (\inf A)^{-1}$.

This page has been left blank for your workings and solutions.