This final exam has ***10 questions*** *on* ***17 pages****, for a total of 90 marks.*

Duration: 2 hours 30 minutes

Section Number (please circle): 201 / 202 / 203

Full Name (including all middle names): ____________________

Student-No: ____________________

Signature: ____________________

UBC Rules governing examinations:

1. Each candidate should be prepared to produce his/her library/AMS card upon request.
2. No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour or the *last 15 minutes* of the examination. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in the examination questions.
3. Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination, and shall be liable to disciplinary action:

   a) Making use of any books, papers or memoranda, other than those authorised by the examiners.

   b) Speaking or communicating with other candidates.

   c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness will not be received.
4. Smoking is not permitted during examinations.

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	10	7	6	7	15	10	7	10	10	8	90
Score:											

Please read the following points carefully before starting to write.

- Give complete arguments and explanations for all your calculations; answers without justifications will not be marked — except where specifically stated.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.). **If you have any of these then tell me now!**
- You may not leave during the first 30 minutes or final 15 minutes of the exam.
- Read all the questions carefully before starting to work.
- Continue on the back of the previous page if you run out of space.

10 marks 1. Give precise mathematical definitions for the following.

(a) Let $f : A \to B$ and let $C \subseteq A$, $D \subseteq B$. Define $f(C)$ and $f^{-1}(D)$.

(b) Let $f : A \to B$ and $g : B \to C$, define the function $g \circ f : A \to C$.

(c) Let S and T be sets, define what it means for S and T to be equinumerous.

(d) Define the Well-Ordering Property of $\mathbb{N}$.

(e) Define the Completeness Axiom of $\mathbb{R}$.

(f) Define the Archimedean Property of $\mathbb{N}$ in $\mathbb{R}$.

(g) Let (a_n) be a sequence, define what it means for (a_n) to converge.

(h) Let (a_n) be a sequence, define what it means for (a_n) to diverge to $-\infty$.

(i) Let (a_n) be a sequence, define what it means for (a_n) to be bounded.

(j) Let (a_n) be a sequence, define what it means for the series $\displaystyle\sum_{n=1}^{\infty} a_n$ to converge.

$\boxed{\text{7 marks}}$ 2. (a) Suppose that $f : A \to B$ and $g : B \to C$. Show that if f and g are surjective then $g \circ f : A \to C$ is surjective.

(b) Suppose that $f : A \to B$ and C_1, C_2 are subsets of A. Show that if f is injective, then $f(C_1 \cap C_2) = f(C_1) \cap f(C_2)$.

6 marks 3. Determine which of the following relations, R, are equivalence relations on the given set A. Prove your answers.

(a) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x^2\}$ on $A = \mathbb{R}$.

(b) $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1)\}$ on $A = \{1, 2, 3\}$.

(c) $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : 3 \text{ divides } a - b\}$ on $A = \mathbb{Z}$.

$\boxed{\text{7 marks}}$ 4. (a) Compute the power set of $\{1, 2, a\}$.

(b) Suppose that S and T are sets, define precisely what $|S| < |T|$ means.

(c) Let $A = (-2, -1] \cup (1, 2)$ and $B = (0, 1)$. Prove that $|A| \leq |B|$.

15 marks

5. Use induction to prove the following.

   (a) Suppose that $x > -1$. Show that $(1+x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.

   (b) Show that $\displaystyle\sum_{n=1}^{N} \frac{1}{n(n+2)} = \frac{3}{4} - \frac{1}{2(N+1)} - \frac{1}{2(N+2)}$ for all integers $N \geq 2$.

   (c) Show that $n^3 - n$ is divisible by 6 for all integers $n \geq 2$.

This page has been left blank for your workings and solutions.

$\boxed{\text{10 marks}}$ 6. (a) Compute the supremum and maximum of the following subsets of $\mathbb{R}$ if they exist. If they do not exist, then write "none". You do not need to justify your answers.

(i) $\{x \in \mathbb{R} \text{ such that } -3 < x \leq 5\}$ (ii) $\displaystyle\bigcup_{n=1}^{\infty}[-1/n, 3 - 1/n^2]$

(iii) $\{x \in \mathbb{Q} \text{ such that } x^2 \leq 2\}$ (iv) $\{x \in \mathbb{R} \text{ such that } \sin(1/x) = 0\}$

(b) Let $k < 0$ and let $f(x) = kx$. Show that if A is a nonempty bounded subset of $\mathbb{R}$ then $\sup f(A) = f(\inf A)$.

7 marks 7. (a) Let $a_n = \dfrac{2n^2 + n + 14}{2n^2 + 11}$. Show, using the definition of convergence, that $a_n \to 1$

(b) Let (a_n) be an increasing sequence and let $k \in \mathbb{R}$. Prove that (ka_n) is a monotone sequence.

10 marks 8. (a) Let α be a non-zero real number and let (s_n) be a sequence. Prove, from the definition of convergence, that if $\lim s_n = s$ then $\lim(\alpha s_n) = \alpha s$.

(b) Let $k \in \mathbb{N}$, and let (s_n) be a sequence. Prove, from the definition of convergence, that if $\lim s_n = s$ then $\lim s_{n+k} = s$.

(c) Fix $k \in \mathbb{N}$, and consider the sequence $s_n = n^{-k}$. Prove, from the definition of convergence, that $s_n \to 0$.

(d) Use (a), (b) and (c) to show that $7/(n+8)^4 \to 0$.

10 marks 9. (a) Do the following sequences converge or diverge? Use the definition of convergence to prove your answer.

(i) $s_n = \frac{1}{n}$ for n odd, $s_n = \frac{3}{n^2}$ for n even.

(ii) $s_n = \cos(n\pi)$.

(b) Suppose that (s_n) is a sequence such that $\lim s_n = s$ with $s > 0$. Prove or disprove that there exists $N \in \mathbb{R}$ such that if $n > N$ then $s_n > 0$.

$\boxed{\text{8 marks}}$ 10. (a) Let (a_n) be the infinite sequence defined by

$$a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 2^{-n} & \text{if } n \text{ is even} \end{cases}$$

Write out the n^{th} partial sum and hence evaluate the series $\sum_{k=1}^{\infty} a_k$.
You may assume the standard result on the convergence of r^n for $r \in \mathbb{R}$.

(b) Let (a_n) be a sequence, and suppose that $a_n \to 2$.

Prove, from the definitions, that $\displaystyle\sum_{n=1}^{\infty} a_n = +\infty$.