Math 217 Final Exam, December 2013

Last Name: ________________ First Name: ________________

Student Number: ________________ Signature: ________________

Instructions. The exam lasts 2.5 hours. **No calculators or electronic devices of any kind are permitted**. There are **10 pages** in this test including this cover page and blank pages. Unless otherwise indicated, show all your work.

Problem #	Value	Grade
1	7	
2	8	
3	8	
4	10	
5	9	
6	8	
Total	50	

1. Consider the function $f(x, y) = e^{xy} + 2x\cos(y)$.

    (a) Compute its first and second partials: f_x, f_y, f_{xx}, f_{xy}, f_{yy} and f_{yx}.

    (b) Find an equation for the tangent plane to its graph at $(x, y) = (1, 0)$.

    (c) Use the linear approximation at $(x, y) = (1, 0)$ to estimate $f(1.02, 0.02)$.

2. Find the centroid (the centre of mass when the density is constant) of the solid region bounded by the surfaces $z = (x^2 + y^2)^2$ and $z = 2 - x^2 - y^2$.

3. Let E be the solid region bounded by the planes $x = 0$, $y = 0$, $x = 1$, and $y = 1$. Let C be the curve of intersection of the surface $z = y^2$ with the boundary of E, oriented counterclockwise when viewed from above. Let

$$\mathbf{F}(x, y, z) = (x - z^3)\hat{\mathbf{i}} + (3xz^2 + y)\hat{\mathbf{j}} + (y + z)\hat{\mathbf{k}}.$$

Use Stokes' theorem to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

4. A rectangular box has two opposing sides (left and right) made of gold, two (front and back) of silver, and two (top and bottom) of bronze. The cost (per unit area) of the gold is \$4/m^2, the silver is \$2/m^2, and the bronze is \$1/m^2. The box has volume 8 m^3, and each of its edges is no more than 8 m long. What are the minimum and maximum possible costs of such a box?

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5. Consider the 2D vector field

$$\mathbf{F}(x, y) = \frac{-y}{x^2 + y^2}\hat{\mathbf{i}} + \frac{x}{x^2 + y^2}\hat{\mathbf{j}} = P(x, y)\hat{\mathbf{i}} + Q(x, y)\hat{\mathbf{j}}$$

defined on $D = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \neq (0, 0)\}$.

(a) Let C_1 be the square of side length 2 centred around the origin (i.e. with corners $(\pm 1, \pm 1)$), with counter-clockwise orientation. Compute the line integral $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$. *Hint: it may help to recall* $\int \frac{du}{1+u^2} = \tan^{-1}(u) + C$.

(b) Let C_2 be the circle of radius 2, $x^2 + y^2 = 4$, with counter-clockwise orientation. Compute the line integral $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

(c) Now you are given that $Q_x(x, y) = P_y(x, y)$ on D (it is true – you do not have to check it!). Your answers to (a) and (b) should be the same; explain why, using Green's theorem.

(d) Is $\mathbf{F}$ conservative on D? If not, explain why not, given that $Q_x = P_y$. If so, find a potential function.

(e) Is $\mathbf{F}$ conservative on the disk of radius 1 centred at $(2, 2)$? If not, explain why not, given that $Q_x = P_y$. If so, find a potential function.

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6. Let S be the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ (a, b, c are positive constants).

(a) Find a parameterization of S, and use it to express the surface integral

$$\int\!\!\int_S \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{1/2} dS$$

as a double integral (but do not try to evaluate the double integral).

(b) Evaluate the surface integral above by using the Divergence Theorem.

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