This examination has 3 pages.

The University of British Columbia

Final Examination – 12 December 2006

Mathematics 217

Multivariable and Vector Calculus

Closed book examination | Time: 150 minutes

Special Instructions: To receive full credit, all answers must be supported with clear and correct derivations. No calculators, notes, or other aids are allowed. A formula sheet is provided with the test.

[12] **1.** A laser fired from the origin strikes the point $P(1,1,3)$ on the mirrored surface

$$z = 6 - (x-2)^2 - 2(y-2)^2.$$

Find the point where the reflected beam strikes the plane $z = 6$.

Hint: The component of the incident beam direction that is normal to the mirror gets reversed by reflection; the component parallel to the mirror is unchanged.

[13] **2.** Astronaut Alpha patrols Sector Zero, the plane region $x > 0$, monitoring Cosmic Disorder (CD). The true CD density at point (x,y) is given by a function Alpha does not know, namely,

$$f(x,y) = x^2 y e^{-x^2-2y^2}.$$

However, Alpha's ship carries instruments that measure $f(x,y)$ and $\nabla f(x,y)$ when it is at (x,y).

(a) Find and classify the critical points of f in Sector Zero.

(b) Alpha flies a mission where the ship's coordinates at time t are given by

$$x = \cos(t), \quad y = 2\sin(t), \quad t \geq 0.$$

As Alpha passes through the point $P(\frac{1}{2}, \sqrt{3})$, does the on-board CD detector indicate that CD is increasing or decreasing? At what rate?

(c) What direction should Alpha fly from P to maximize the instantaneous rate of increase in CD? What is the angle between this direction and the line from P to the point of maximum CD? (*Note*: A calculator-ready numerical expression for the cosine of the requested angle is fully acceptable.)

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[13] **3.** Given $a > 0$, consider the triangle D whose vertices are $(0,0)$, $(0,a)$, $(2a,a)$. Define a surface S by

$$z = 1 + 3x + 2y^2.$$

(a) Find the plane tangent to S at the point where $x = a$ and $y = a$.

(b) Find the area of the part of the tangent plane that lies above D. Call this area $\beta(a)$.

(c) Find the area of the part of the surface S that lies above D. Call this area $\gamma(a)$.

(d) [OPTIONAL BONUS QUESTION] Prove: $\lim_{a\to 0^+} \dfrac{\gamma(a)}{\beta(a)} = 1.$

[12] **4.** Evaluate $I = \displaystyle\int_0^9 \int_0^{\sqrt{y}} y e^{27x - x^3}\, dx\, dy + \int_9^{18} \int_0^{\sqrt{18-y}} y e^{27x-x^3}\, dx\, dy.$

[13] **5.** Let C denote the closed loop in which the cylinder $x^2 + y^2 = 2ax$ meets the plane $z = y$. Given

$$\mathbf{F} = y^2\,\mathbf{i} + \tan^{-1} z\,\mathbf{j} + (1 + x^2)\,\mathbf{k}, \qquad a > 0,$$

find the work done by $\mathbf{F}$ acting around C. Orient C counterclockwise when viewed from above.

[12] **6.** Let $\mathbf{F}(x,y,z) = \big(y^2 \cos z\big)\,\mathbf{i} + \big(-xy^2 \sin z\big)\,\mathbf{k}.$

(a) Prove that $\mathbf{F}$ is not conservative.

(b) Find a scalar field $Q = Q(x,y,z)$ such that $\mathbf{G}$ is conservative, where

$$\mathbf{G}(x,y,z) = \mathbf{F}(x,y,z) + Q(x,y,z)\mathbf{j}.$$

(c) Find $W = \displaystyle\int_C \mathbf{F} \bullet d\mathbf{r}$, given C: $\quad x = \cos t, \quad y = \sin t, \quad z = t, \qquad 0 \le t \le \pi/2.$

[13] **7.** Evaluate the flux $I = \displaystyle\iint_S \mathbf{F} \bullet d\mathbf{S}$ in each of the situations below.

(a) S is the boundary surface for the solid cylinder $E = \big\{(x,y,z) : x^2 + y^2 \le a^2,\ 0 \le z \le H\big\}$, and

$$\mathbf{F}(x,y,z) = \big\langle -yze^{xyz}, xze^{xyz}, xy^2e^z + z^2\big\rangle.$$

(b) $S = \big\{(x,y,z) : 0 \le z = 9 - x^2 - y^2\big\}$ and $\mathbf{F} = \langle xy,\ yz,\ xz\rangle$. (Use upward orientation on S.)

(c) S is the two-part surface with bottom $z = \sqrt{x^2 + y^2}$ and top $z = \sqrt{a^2 - x^2 - y^2}$, and

$$\mathbf{F} = \big\langle xy^2,\ yz^2,\ zx^2\big\rangle.$$

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[12] **8.** Suppose the following equations parametrize a smooth closed curve C in the plane $z = H$:

$$x = f(u), \quad y = g(u), \quad z = H, \qquad a \le u \le b.$$

Assume that $H > 0$, and that the parametrization gives a counterclockwise direction of motion around C when viewed from above. Let D denote the plane region enclosed by the curve C.

(a) Write a single integral with respect to u that returns the area of D. Call this area A.

(b) Let S denote the surface generated by all line segments joining the origin to a point on the curve C. Parametrize S.

(c) Prove that the flux of $\mathbf{F}(x, y, z) = w(x, y, z)\,(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ across S equals zero for every smooth scalar field w.

(d) Let E denote the solid enclosed by the plane region D and the surface S; write $V = \text{Vol}(E)$. Prove:

$$V = \frac{1}{3}AH.$$

[*Hint*: Find a way to use the vector field in part (c) with $w(x, y, z) = 1$.]

[100] **Total Marks**

The End

MATH 217 FORMULAS FOR FINAL EXAMINATION, 12 DECEMBER 2006

VECTOR IDENTITIES

For $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$,

$\mathbf{u} \bullet \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = |\mathbf{u}|\,|\mathbf{v}|\cos(\theta), \quad 0 \le \theta \le \pi$

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

Length of $\mathbf{u}$: $|\mathbf{u}| = \sqrt{\mathbf{u}\bullet\mathbf{u}} = \sqrt{u_1^2 + u_2^2 + u_3^2}$

Angle between $\mathbf{u}$ and $\mathbf{v}$: $\theta = \cos^{-1}\left(\dfrac{\mathbf{u}\bullet\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right), \quad 0 \le \theta \le \pi$

Triple product identities: $\mathbf{u}\bullet(\mathbf{v}\times\mathbf{w}) = \mathbf{v}\bullet(\mathbf{w}\times\mathbf{u}) = \mathbf{w}\bullet(\mathbf{u}\times\mathbf{v})$ $\qquad$ $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\bullet\mathbf{w})\mathbf{v} - (\mathbf{u}\bullet\mathbf{v})\mathbf{w}$

DISTANCES AND PROJECTIONS

Point (x_0, y_0, z_0) to plane $Ax + By + Cz = D$: $s = \dfrac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}$ $\qquad$ $\mathbf{F} = \mathbf{proj}_\mathbf{u}(\mathbf{F}) + \mathbf{orth}_\mathbf{u}(\mathbf{F})$

Point $\mathbf{r}_0 = (x_0, y_0, z_0)$ to line $\mathbf{r} = \mathbf{r}_1 + t\mathbf{v}$: $s = \dfrac{|(\mathbf{r}_0 - \mathbf{r}_1)\times\mathbf{v}|}{|\mathbf{v}|}$ $\qquad$ $\mathbf{proj}_\mathbf{u}(\mathbf{F}) = \left(\dfrac{\mathbf{F}\bullet\mathbf{u}}{\mathbf{u}\bullet\mathbf{u}}\right)\mathbf{u}$

Line $\mathbf{r} = \mathbf{r}_1 + t\mathbf{v}_1$ to line $\mathbf{r} = \mathbf{r}_2 + t\mathbf{v}_2$: $s = \dfrac{|(\mathbf{r}_2 - \mathbf{r}_1)\bullet(\mathbf{v}_1\times\mathbf{v}_2)|}{|\mathbf{v}_1\times\mathbf{v}_2|}$ $\qquad$ $\mathbf{orth}_\mathbf{u}(\mathbf{F}) = \mathbf{F} - \mathbf{proj}_\mathbf{u}(\mathbf{F}) = \dfrac{(\mathbf{u}\bullet\mathbf{u})\mathbf{F} - (\mathbf{F}\bullet\mathbf{u})\mathbf{u}}{\mathbf{u}\bullet\mathbf{u}}$

VECTOR-VALUED FUNCTIONS OF ONE VARIABLE

$\dfrac{d}{dt}(\lambda(t)\mathbf{u}(t)) = \lambda'(t)\mathbf{u}(t) + \lambda(t)\mathbf{u}'(t)$ $\qquad$ $\dfrac{d}{dt}(\mathbf{u}(t)\bullet\mathbf{v}(t)) = \mathbf{u}'(t)\bullet\mathbf{v}(t) + \mathbf{u}(t)\bullet\mathbf{v}'(t)$ $\qquad$ $\dfrac{d}{dt}(\mathbf{u}(t)\times\mathbf{v}(t)) = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$

$\dfrac{d}{dt}(\mathbf{u}(\lambda(t))) = \lambda'(t)\mathbf{u}'(\lambda(t))$ $\qquad$ $\dfrac{d}{dt}|\mathbf{u}(t)| = \dfrac{\mathbf{u}(t)\bullet\mathbf{u}'(t)}{|\mathbf{u}(t)|}, \quad \mathbf{u}(t) \ne \mathbf{0}$

Position $\mathbf{r} = \mathbf{r}(t)$ gives velocity $\mathbf{v}(t) = \mathbf{r}'(t)$, speed $v(t) = |\mathbf{v}(t)|$, acceleration $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = \left(\dfrac{dv}{dt}\right)\widehat{\mathbf{T}} + \dfrac{v^2}{\rho}\widehat{\mathbf{N}}$; $\qquad$ $\widehat{\mathbf{T}} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$

$ds = v(t)\,dt = |\mathbf{v}(t)|\,dt = \left|\dfrac{d\mathbf{r}}{dt}\right|dt = |d\mathbf{r}|$ $\qquad$ $d\mathbf{r} = \dfrac{d\mathbf{r}}{dt}\,dt = \mathbf{v}(t)\,dt = \left\langle \dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dz}{dt}\right\rangle dt = \langle dx, dy, dz\rangle\,dt$

APPROXIMATIONS

Differentiability test for scalar field f at $\mathbf{a}$: $\quad 0 = \lim\limits_{\mathbf{x}\to\mathbf{a}} \dfrac{E(\mathbf{x})}{|\mathbf{x}-\mathbf{a}|}$, where $E(\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{a}) - \nabla f(\mathbf{a})\bullet(\mathbf{x}-\mathbf{a})$

Tangent Hyperplane for $G(\mathbf{x}) = 0$ at $\mathbf{a}$: $\quad 0 = \nabla G(\mathbf{a})\bullet(\mathbf{x}-\mathbf{a})$ $\qquad$ (a line in $\mathbb{R}^2$; a plane in $\mathbb{R}^3$; a hyperplane in $\mathbb{R}^n$)

Linearization of f around $\mathbf{a}$: $\quad f(\mathbf{x}) \approx L(\mathbf{x})$ for $\mathbf{x} \approx \mathbf{a}$, where $\quad L(\mathbf{x}) = f(\mathbf{a}) + (\mathbf{x}-\mathbf{a})\bullet\nabla f(\mathbf{a})$

Differentials (case $\mathbf{x}\in\mathbb{R}^3$): $\quad df = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy + \dfrac{\partial f}{\partial z}dz = \nabla f\bullet d\mathbf{r}$ $\qquad$ $\Delta f \approx \dfrac{\partial f}{\partial x}\Delta x + \dfrac{\partial f}{\partial y}\Delta y + \dfrac{\partial f}{\partial z}\Delta z = \nabla f\bullet\Delta\mathbf{r}$

Quadratic Approx, for $(x,y)\in\mathbb{R}^2$ near (a,b): $\quad f(x,y) \approx Q(x,y) = f(a,b) + f_1(a,b)(x-a) + f_2(a,b)(y-b)$

$$+\tfrac{1}{2}\Big[f_{11}(a,b)(x-a)^2 + f_{22}(a,b)(y-b)^2 + 2f_{12}(a,b)(x-a)(y-b)\Big]$$

SECOND DERIVATIVE TEST FOR (a,b) WHERE $\nabla f(a,b) = (0,0)$

$H(x,y) = \begin{bmatrix} f_{11}(x,y) & f_{12}(x,y) \\ f_{21}(x,y) & f_{22}(x,y)\end{bmatrix}$ $\qquad$ $H(a,b) = \begin{bmatrix} A & B \\ B & D\end{bmatrix}$ $\qquad$ $\Delta = \det(H(a,b)) = AD - B^2$

$\Delta < 0 \implies$ saddle $\qquad$ $\Delta > 0, A > 0 \implies$ loc min $\qquad$ $\Delta > 0, A < 0 \implies$ loc max

DERIVATIVES

$\nabla = \mathbf{i}\dfrac{\partial}{\partial x} + \mathbf{j}\dfrac{\partial}{\partial y} + \mathbf{k}\dfrac{\partial}{\partial z} = \left\langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}\right\rangle$ $\qquad$ $\mathbf{F}(x,y,z) = F_1(x,y,z)\,\mathbf{i} + F_2(x,y,z)\,\mathbf{j} + F_3(x,y,z)\,\mathbf{k}$

$\nabla\phi(x,y,z) = \mathbf{grad}\,\phi(x,y,z) = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k}$ $\qquad$ $\nabla\bullet\mathbf{F}(x,y,z) = \mathbf{div}\ \mathbf{F}(x,y,z) = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$

$$\nabla\times\mathbf{F}(x,y,z) = \mathbf{curl}\,\mathbf{F}(x,y,z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3\end{vmatrix} = \left(\dfrac{\partial F_3}{\partial y} - \dfrac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial F_1}{\partial z} - \dfrac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}\right)\mathbf{k}$$

$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$ $\qquad$ $\nabla\bullet(\mathbf{F}\times\mathbf{G}) = (\nabla\times\mathbf{F})\bullet\mathbf{G} - \mathbf{F}\bullet(\nabla\times\mathbf{G})$

$\nabla\bullet(\phi\mathbf{F}) = (\nabla\phi)\bullet\mathbf{F} + \phi\,(\nabla\bullet\mathbf{F})$ $\qquad$ $\nabla\times(\mathbf{F}\times\mathbf{G}) = \mathbf{F}(\nabla\bullet\mathbf{G}) - \mathbf{G}\,(\nabla\bullet\mathbf{F}) - (\mathbf{F}\bullet\nabla)\mathbf{G} + (\mathbf{G}\bullet\nabla)\mathbf{F}$

$\nabla\times(\phi\mathbf{F}) = (\nabla\phi)\times\mathbf{F} + \phi\,(\nabla\times\mathbf{F})$ $\qquad$ $\nabla(\mathbf{F}\bullet\mathbf{G}) = \mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{G}\times(\nabla\times\mathbf{F}) + (\mathbf{F}\bullet\nabla)\mathbf{G} + (\mathbf{G}\bullet\nabla)\mathbf{F}$

$\nabla\times(\nabla\phi) = \mathbf{0}$ $\quad$ (**curl grad** $= \mathbf{0}$) $\qquad$ $\nabla\bullet(\nabla\times\mathbf{F}) = 0$ $\quad$ (**div curl** $= 0$)

$\nabla^2\phi(x,y,z) = \nabla\bullet\nabla\phi(x,y,z) = \mathbf{div}\,\mathbf{grad}\,\phi = \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2}$ $\qquad$ $\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\bullet\mathbf{F}) - \nabla^2\mathbf{F}$ $\quad$ (**curl curl** $=$ **grad div** $-$ laplacian)

SURFACE NORMALS AND AREA ELEMENTS

$\mathbf{r} = \mathbf{r}(u, v)$ (parametrized surface): $\mathbf{n} = \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)$; $\widehat{\mathbf{N}} = \pm\frac{\mathbf{n}}{|\mathbf{n}|}$ $d\mathbf{S} = \pm\left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) du\, dv$

$G(x, y, z) = 0$ (smooth level surface): $\mathbf{n} = \nabla G(x, y, z)$; $\widehat{\mathbf{N}} = \pm\frac{\mathbf{n}}{|\mathbf{n}|}$ $d\mathbf{S} = \pm\frac{\nabla G(x, y, z)}{|\partial G/\partial z|}\, dx\, dy$

$d\mathbf{S} = \frac{\mathbf{n}}{|\mathbf{n} \bullet \mathbf{k}|}\, dx\, dy = \frac{\mathbf{n}}{|\mathbf{n} \bullet \mathbf{j}|}\, dx\, dz = \frac{\mathbf{n}}{|\mathbf{n} \bullet \mathbf{i}|}\, dy\, dz$ for other projections $d\mathbf{S} = \widehat{\mathbf{N}}\, dS$; $dS = |d\mathbf{S}|$

POLAR AND CYLINDRICAL COORDINATES

Transformation: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

Position vector: $\mathbf{r} = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + z\mathbf{k}$

Volume element: $dV = r\, dr\, d\theta\, dz$

Surface area element (on $r = a$): $dS = a\, d\theta\, dz$

Surface area element (on $z = 0$): $dS = r\, dr\, d\theta$

SPHERICAL COORDINATES

Transformation: $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$

Position vector: $\mathbf{r} = \rho\sin\phi\cos\theta\,\mathbf{i} + \rho\sin\phi\sin\theta\,\mathbf{j} + \rho\cos\phi\mathbf{k}$

Volume element: $dV = \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

Surface area element (on $\rho = a$): $dS = a^2\sin\phi\, d\theta\, d\phi$

INTEGRATING DERIVATIVES: THE FUNDAMENTAL THEOREM OF CALCULUS

$\int_a^b f'(t)\, dt = f(b) - f(a)$ (the one-dimensional **Fundamental Theorem**)

$\int_C \nabla\phi \bullet d\mathbf{r} = \phi(\mathbf{r}(b)) - \phi(\mathbf{r}(a))$, if C is the curve $\mathbf{r} = \mathbf{r}(t)$, $a \le t \le b$

$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$

$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_{\partial D} \mathbf{F} \bullet d\mathbf{r} = \oint_{\partial D} P(x, y)\, dx + Q(x, y)\, dy$, where ∂D is the positively oriented boundary of D **(Green's Theorem)**

$\iint_S \nabla \times \mathbf{F} \bullet \widehat{\mathbf{N}}\, dS = \oint_{\partial S} \mathbf{F} \bullet d\mathbf{r} = \oint_{\partial S} P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz$, where ∂S is the oriented boundary of S **(Stokes's Theorem)**

$\iiint_E \nabla \bullet \mathbf{F}\, dV = \oiint_{\partial E} \mathbf{F} \bullet \widehat{\mathbf{N}}\, dS$, where ∂E is the closed boundary of E, with outward unit normal $\widehat{\mathbf{N}}$ **(Divergence Theorem)**

AVERAGE VALUES FOR FUNCTION f ON CURVE $\mathcal{C}$, FUNCTION g ON SURFACE $\mathcal{S}$, FUNCTION h ON SOLID E

$\overline{f} = \frac{\int_{\mathcal{C}} f\, ds}{\int_{\mathcal{C}} 1\, ds}$ $\overline{g} = \frac{\iint_{\mathcal{S}} g\, dS}{\iint_{\mathcal{S}} 1\, dS}$ $\overline{h} = \frac{\iiint_E h(x, y, z)\, dV}{\iiint_E 1\, dV}$

SINGLE INTEGRALS

$\int x\sin(bx)\, dx = \frac{\sin(bx)}{b^2} - \frac{x\cos(bx)}{b}$ $\int x\cos(bx)\, dx = \frac{\cos(bx)}{b^2} + \frac{x\sin(bx)}{b}$ $\int xe^{ax}\, dx = \frac{e^{ax}}{a^2}(ax - 1)$

$\int e^{ax}\sin(bx)\, dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2 + b^2}$ $\int e^{ax}\cos(bx)\, dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2 + b^2}$ $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

$\int \sec^2 x\, dx = \tan x$ $\int \sin^2 x\, dx = \frac{x}{2} - \frac{1}{4}\sin 2x$ $\int \cos^2 x\, dx = \frac{x}{2} + \frac{1}{4}\sin 2x + C$

$\int \tan x\, dx = \ln|\sec x|$ $\int \sin^3 x\, dx = \frac{1}{3}\cos^3 x - \cos x$ $\int \cos^3 x\, dx = \sin x - \frac{1}{3}\sin^3 x$

$\int \tan^2 x\, dx = \tan x - x$ $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$ $(a > 0)$ $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{x + a}{x - a}\right|$, $(a > 0)$

$\int \frac{dx}{(a^2 - x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2 - x^2}}$ $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$ $(a > 0)$ $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

$\int \frac{dx}{(x^2 \pm a^2)^{3/2}} = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}}$ $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$ $\int \sqrt{x^2 \pm a^2}\, dx = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2}\ln\left|x + \sqrt{x^2 \pm a^2}\right|$

$\int_0^{\pi/2} \sin x\, dx = \int_0^{\pi/2} \cos x\, dx = 1$ $\int_0^{\pi/2} \sin^2 x\, dx = \int_0^{\pi/2} \cos^2 x\, dx = \frac{\pi}{4}$ $\int_0^{\pi/2} \sin^3 x\, dx = \int_0^{\pi/2} \cos^3 x\, dx = \frac{2}{3}$

$\int_0^{\pi/2} \sin^4 x\, dx = \int_0^{\pi/2} \cos^4 x\, dx = \frac{3\pi}{16}$ $\int_0^{\pi/2} \sin^5 x\, dx = \int_0^{\pi/2} \cos^5 x\, dx = \frac{8}{15}$ $\int_0^{\pi/2} \sin^6 x\, dx = \int_0^{\pi/2} \cos^6 x\, dx = \frac{5\pi}{32}$

TRIGONOMETRIC IDENTITIES

$\sin^2 x + \cos^2 x = 1$ $\sin(-x) = -\sin x$ $\cos(-x) = \cos x$

$\sec^2 x = 1 + \tan^2 x$ $\csc^2 x = 1 + \cot^2 x$ $\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x\tan y}$

$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$ $\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$ $\sin\left(\frac{\pi}{2}\right) = 1 = \cos(0)$

$\sin^2 x = \frac{1 - \cos 2x}{2}$ $\cos^2 x = \frac{1 + \cos 2x}{2}$ $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = \cos\left(\frac{\pi}{4}\right)$

$\sin(0) = 0 = \cos\left(\frac{\pi}{2}\right)$ $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right)$ $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right)$

Adapted from R. A. Adams, *Calculus, A Complete Course,* Addison-Wesley, 2003.