THE UNIVERSITY OF BRITISH COLUMBIA

Sessional Examinations April 2015

MATH 215

Time: 2.5 hours

No aids permitted. A table of Laplace transforms is supplied on the last page.

1. (a) Solve the initial value problem

$$t^2y' + 2ty = \cos(\pi t) \qquad y(1) = 1$$

for $y(t)$.

(b) Solve the initial value problem

$$y' = t + y^2 t \qquad y(0) = \alpha.$$

for $y(t)$. What is the largest interval of t values on which the solution exists?

(c) Is there a value of α in the initial value problem of part (b) so that the solution $y(t)$ exists for all t? Give a reason.

2. The volume V of an evaporating spherical water drop changes at a rate proportional to its surface area so that

$$V' = -kV^{2/3}$$

for some $k > 0$.

(a) If a drop with initial volume $V(0) = 1$ evaporates completely when $t = 1$, when will a drop with initial volume $V(0) = 1/2$ evaporate completely?

(b) The two solutions in part (a) to the equation for V (with initial values 1 and 1/2) are both equal to zero when $t = 1$. Why does this not contradict Picard's theorem on existence and uniqueness for the first order differential equations?

3. (a) Solve

$$y'' - 6y' + 8y = 0 \quad \text{with} \quad y(0) = 0, y'(0) = 1.$$

(b) Solve

$$y'' + 2y' + 5y = 0 \quad \text{with} \quad y(0) = 1, y'(0) = 0.$$

4. Each of the following equations has a particular solution that can be found by the method of undetermined coefficients (or judicious guessing). Write down the *form* of the solution in each case. (You do not have to actually find the solutions.) Equations (d), (e) and (f) can be solved either by guessing a real form, or by guessing the solution to a related complex equation and then taking the real part. You may give either guess.

(a) $y'' - 6y' + 8y = 1$

(b) $y'' - 6y' + 8y = e^t$

(c) $y'' - 6y' + 8y = e^{2t}$

(d) $y'' + 2y' + 5y = t^2 \cos(2t)$

(e) $y'' + 2y' + 5y = t^2 e^t \cos(2t)$

(f) $y'' + 2y' + 5y = \cos^2(t)$

5. The equation

$$x'' + 4x = \alpha\delta(t - t_0) \quad \text{with} \quad x(0) = 0, x'(0) = 2$$

models a frictionless mass-spring system that is hit with a hammer at time $t = 0$ and again at $t = t_0$.

(a) Find the Laplace tranform $X(s)$ of $x(t)$.

(b) Invert the Lapace transform to determine $x(t)$.

(c) Find values of α and t_0 so that the mass is stationary after time t_0.

6. (a) Find two independent real solutions to the linear homogeneous system

$$\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \mathbf{x}$$

(b) Solve the initial value problem

$$\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

7. Suppose the populations $x(t)$ and $y(t)$ of two competing species can be modeled by the autonomous system

$$x' = (4 - x - y)x$$
$$y' = (6 - x - 2y)y$$

(a) Determine the equilibrium points.

(b) Draw the nullclines, i.e., the lines in the x, y plane where $x' = 0$ and the lines where $y' = 0$. Indicate the direction of the vector field along the segments of the nullclines between equilibrium points.

(c) For each equilibrium point, write down the approximating linear system. Determine which equilibrium points (if any) are stable, unstable or saddle points.

(d) Draw a qualitatively accurate phase portrait of the system. Can these species co-exist? Explain. Under what initial conditions will only one species be present in the limit $t \to \infty$.

[add a table of laplace tranforms]