University of British Columbia
Math 215/255, Sections 101 (Rahmani), 102 (Froese), 103 (Yang) and 104 (Daskalakis)
Final Exam, December 2016

Name (print): ____________________

Student ID Number: ____________ **Signature:** ____________

Rules governing examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  (a) speaking or communicating with other candidates, unless otherwise authorized;
  (b) purposely exposing written papers to the view of other candidates or imaging devices;
  (c) purposely viewing the written papers of other candidates;
  (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Question	Points	Score
1	0	
2	0	
3	0	
4	0	
5	0	
6	0	
7	0	
8	0	
Total:	0	

Additional Instructions:

- No notes, books or calculators are allowed.
- Read the questions carefully and make sure you provide all the information that is asked for in the question.
- Show all your work. Correct answers without explanation or accompanying work could receive no credit.
- Answer the questions in the space provided. Continue on the back of the page if necessary.

1. A tank with a capacity of 1000 liters contains 500 liters of pure water. Brine containing 0.1 kg of salt per liter enters the tank at 10 liters per minute. The contents of the tank are kept thoroughly mixed. The mixed solution drains from the bottom tank at 8 liters per minute and overflows at the top when the capacity is reached.

   (a) What is the volume $V(t)$ of the water in the tank at time t minutes for $0 \le t < \infty$? At what time T does the tank start overflowing?

   Solution: When $t \le T$, the volume in the tank is increasing at 2 liters/minute, so $T = 500/2 = 250$ minutes, and

$$V(t) = \begin{cases} 500 + 2t & 0 \le t \le 250 \\ 1000 & t \ge 250. \end{cases}$$

   (b) Let $S(t)$ be the amount (in kg) of salt in the tank at time t. Write down the differential equation satisfied by $S(t)$. (Hint: consider $t < T$ and $t > T$ separately.)

   Solution: The rate of increase of salt is $10 \times 0.1 = 1$ kg/minute. The concentration of salt in the tank is $S(t)/V(t)$. The mixture leaves the tank at 8 liters/minute for $t < T$ and 10 liters/minute for $t > T$. Thus

$$S'(t) = \begin{cases} 1 - \frac{8S(t)}{500+2t} & 0 \le t < 250 \\ 1 - \frac{10S(t)}{1000} & t > 250 \end{cases}$$

   (c) Compute $S(t)$ for $0 \le t \le T$.

2. Consider the equation

$$x^2 + 2tx - t^2x' = 0, \tag{1}$$

(a) Find the solution $x = x(t)$ to (1) with the initial condition

$$x(1) = 1.$$

(b) Find the largest interval containing $t = 1$ where $x(t)$ is defined and satisisfies (1).

(c) Can there be other solutions to (1) passing through $(x, t) = (1, 1)$? *Hint: What does the existence and uniqueness theorem for first order equations say?*

(d) At $t = 0$, the solution $x(t)$ from part (a) satisfies the initial condition

$$x(0) = 0.$$

Find another solution to (1) (i.e., different from $x(t)$) that satisfies the same initial condition. Does this contradict the existence and uniqueness theorem? Explain.

Solution: It is easy to show that $x = 0$ is a solution to (1). If $x \neq 0$, then we get

$$\left(1 + \frac{2t}{x}\right)\mathrm{d}t = \frac{t^2}{x^2}\mathrm{d}x.$$

Let $v = t/x$, then $vx = t$. This implies that

$$x\mathrm{d}v + v\mathrm{d}x = \mathrm{d}t,$$

i. e.

$$v\mathrm{d}x = \mathrm{d}t - \frac{t}{v}\mathrm{d}v.$$

By plugging in, we obtain the equation

$$(1 + 2v)\mathrm{d}t = v^2\mathrm{d}x = v\mathrm{d}t - t\mathrm{d}v,$$

which can be written in the form

$$(1 + v)\mathrm{d}t = -t\mathrm{d}v.$$

Hence, the function v can be written in the form

$$v = Ct^{-1} - 1,$$

and then

$$x = tv^{-1} = \frac{t^2}{C - t}.$$

The initial condition implies that

$$1 = \frac{1}{C-1},$$

i. e.

$$C = 2.$$

Whence, the general solution is

$$x = \frac{t^2}{2-t}.$$

3. (a) The functions $y_1(x) = x$ and $y_2(x) = 1/x$ are solutions to the equation

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0, \qquad 0 < x < \infty.$$

Are they linearly independent? Give a reason.

(b) Find a particular solution y_p to

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = x^2, \qquad 0 < x < \infty. \tag{2}$$

(c) Find the solution $y(x)$ of (2) that satisfies the initial condition $y(1) = 1$, $y'(1) = 0$.

4. Consider the equation

$$y'' + (-2 + 2x)y' + (1 - 2x)y = 0. \tag{3}$$

It is easy to show that the function $y_1 = e^x$ is a solution to (3). Use y_1 to find another solution to (3).

5. Find the general solution to

$$x'' + 2x' + 5x = 4e^{-t} + 17\sin(2t). \tag{4}$$

Solution: The characteristic equation is

$$\lambda^2 + 2\lambda + 5 = 0,$$

which will can be solved for

$$\lambda_1 = -1 + 2i, \qquad \lambda_1 = -1 - 2i.$$

The complementary solution is

$$x_c = e^{-t}\big(c_1\cos(2t) + c_2\sin(2t)\big).$$

We now want to find a particular solution in the form

$$x_p = Ae^{-t} + B\cos(2t) + C\sin(2t).$$

Then there hold

$$x_p' = -Ae^{-t} - 2B\sin(2t) + 2C\cos(2t),$$

and

$$x_p'' = Ae^{-t} - 4B\cos(2t) - 4C\sin(2t).$$

By plugging in, we get

$$4Ae^{-t} + (B + 4C)\cos(2t) + (C - 4B)\sin(2t) = 4e^{-t} + 17\sin(2t).$$

This gives that

$$A = 1, \quad B = -4, \quad C = 1.$$

The general solution is

$$x = e^{-t}\big(c_1\cos(2t) + c_2\sin(2t)\big) + e^{-t} - 4\cos(2t) + \sin(2t).$$

6. (a) Find the Laplace transform of the following piecewise continuous function:

$$g(t) = \begin{cases} \sin(t), & 0 \leq t < \pi \\ 0, & t \geq \pi \end{cases}$$

(b) Let $Y(s)$ to be the Laplace transform of the solution $y(t)$ to the following intial value problem:

$$y'' + y' + \frac{5}{4}y = g(t), \qquad y(0) = 0, \qquad y'(0) = 0$$

where $g(t)$ is the function defined in (a). Determine $Y(s)$.

(c) Calculate the inverse Laplace transform of $Y(s)$ to find the solution $y(t)$ [[include a table!]]

Solution:

7. Consider the second order differential equation with initial conditions

$$y'' + 2y' + y = 0, \qquad y(0) = 1, \qquad y'(0) = 1.$$

(a) Convert the equation into a 2×2 system of first order equations with the form $\mathbf{x}' = A\mathbf{x}$ with initial condition $\mathbf{x}(0) = \mathbf{b}$.

(b) Find the general solution to the system in (a) and sketch some typical trajectories in the plane.

(c) Solve the initial value problem for the system in part (a) and use the solution to determine $y(t)$.

Solution:

8. Let $y(t)$ be the solution to the initial value problem

$$y' = t + ty, \qquad y(1) = 1.$$

(a) Use Euler's method with step size $h = 0.1$ to estimate $y(1.2)$.

(b) Now consider the system of equations for $x_1(t)$, $x_2(t)$:

$$\begin{aligned} x_1' &= t + tx_2 \qquad & x_1(1) &= 1 \\ x_2' &= t + tx_1 \qquad & x_2(1) &= 0. \end{aligned}$$

Adapt Euler's method with step size $h = 0.1$ to this situation to obtain an estimate for $x_1(1.2)$ and $x_2(1.2)$.

Solution: