The University of British Columbia

Final Examination - April 12th, 2016

Mathematics 215/255

Time: 2 hours

Last Name ____________________ **First** _____________ **Signature** ____________________

Student Number _____________________

Section (circle one) Coombs (215:201) / Herrera (215:202) / Rahmani (255:201)

Special Instructions:

- Books, notes, cellphones and calculators are **not** allowed.
- Use the reverse side of each page if you need extra space.
- **Show your work**. A correct answer without intermediate steps will receive no credit.
- This exam has 12 pages, including this cover and a table of Laplace transforms.

1		10
2		10
3		15
4		10
5		10
6		5
7		20
8		20
Total		100

Problem 1 (10 Marks) Your lab partner leaves a drop of bleach on the lab bench, which takes the shape of a hemisphere. At $t = 0$ the drop has a radius of 6mm. The drop loses volume due to evaporation at a rate proportional to its surface area. When $t = 10$ minutes, the radius is 4mm. At what time has the drop completely evaporated? Show your work. Note: Only the part of the drop surface that is exposed to the air can evaporate.

The drop is gone when $t =$ minutes.

Problem 2 (10 Marks) a. Solve the initial value problem for $y(t)$. Show your work.

$$y'' + \pi^2 y = 0, \qquad y(0) = 1; \quad y'(0) = 1.$$

$y(t) =$

b. Draw a clear graph of the solution, indicating the period and amplitude. Show your work.

Problem 3 (15 Marks) Find a fundamental matrix $\mathbf{X}(t)$ and determine the general solution to

$$\vec{x}'(t) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \vec{x} + \begin{pmatrix} -1 \\ -e^{-t} \end{pmatrix}.$$

Show your work.

$\mathbf{X}(t) =$
$\vec{x}(t) =$

Problem 4 (10 Marks) Find the inverse Laplace transforms of the indicated functions. Show your work.

(1)

$$F(s) = \frac{2}{s^2 + 3s - 4}$$

(2)

$$G(s) = \frac{2s + 1}{s^2 - 2s + 2}$$

$f(t) =$
$g(t) =$

Problem 5 (10 Marks) Solve the initial value problem using Laplace transform. Answers obtained using any other method will not receive any points. Show your work.

$$y'' - 2y' + 2y = \cos(t) + 2\sin(t), \quad y(0) = 2, \quad y'(0) = 0$$

$y(t) =$

Problem 6 (5 Marks) For what values of α does the solution to the following problem remain bounded as $t \to \infty$? Hint: use your work from the previous problem. Show your work.

$$y'' - 2y' + 2y = \cos(t) + 2\sin(t), \quad y(0) = \alpha, \quad y'(0) = 0$$

The solution remains bounded for

Problem 7 (20 Marks) A stylized model of competing species with population densities $x(t)$ and $y(t)$ is given by

$$\frac{dx}{dt} = x(1 - x - y),$$
$$\frac{dy}{dt} = y\left(\frac{1}{2} - \frac{y}{4} - \frac{3x}{4}\right).$$

This model has critical points at $(0,0), (0,2), (1,0)$ and $(\frac{1}{2}, \frac{1}{2})$.

(1) Find the Jacobian matrix for this system.

$J(t) =$

(2) Classify each critical point. Show your work.

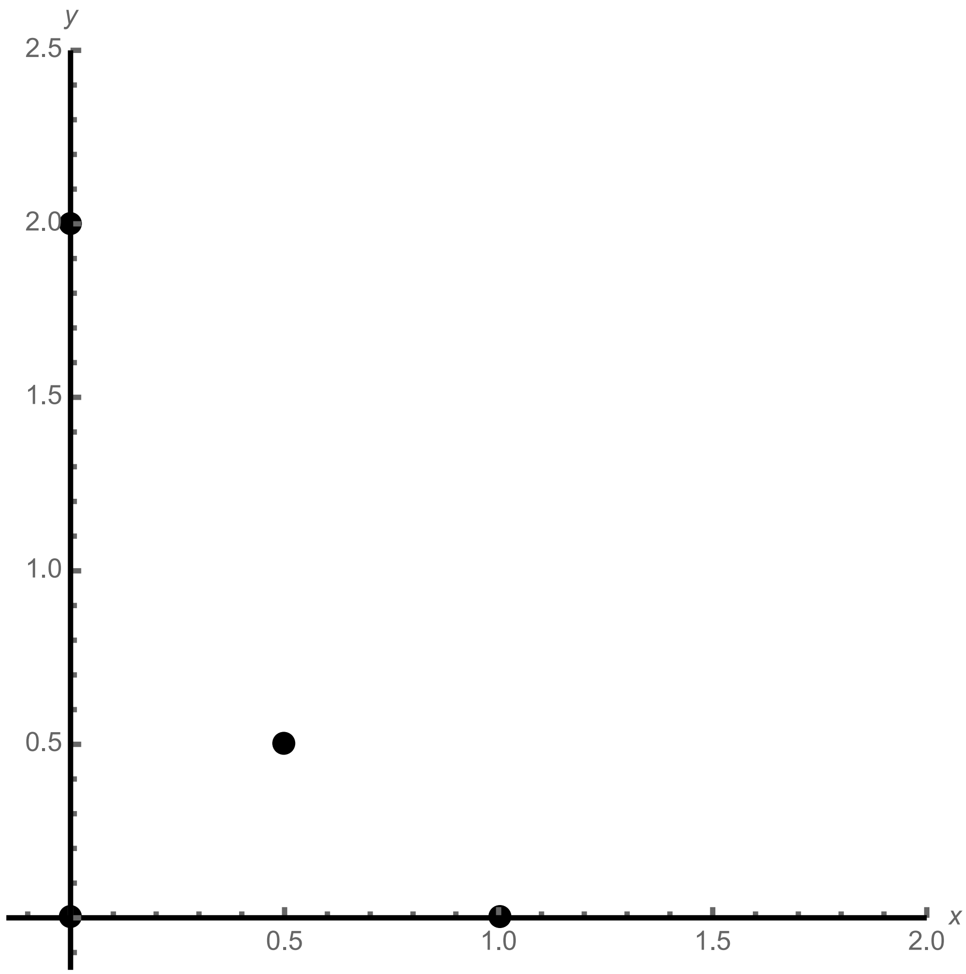
$(x, y) = (0,0)$ is

$(x, y) = (0,2)$ is

$(x, y) = (1,0)$ is

$(x, y) = (\frac{1}{2}, \frac{1}{2})$ is

(3) Sketch representative solutions of the nonlinear system on the axes below:



Problem 8 (20 Marks) Consider a forced second order equation modeling the charge stored in a capacitor, which can be described by:

$$Q''(t) + RQ'(t) + \frac{1}{C}Q = F\cos(\omega t), \quad Q(0) = 0, \quad Q'(0) = 1,$$

R and C are parameters and the forcing has amplitude F and frequency ω.

On the following page you will find plots of different solutions $Q(t)$ for various values of R, C, F, and ω, corresponding in no particular order to:

A. $Q''(t) + 64Q(t) = \cos(8t)$.

B. $Q''(t) + 0.8Q'(t) + 64Q(t) = 0$.

C. $Q''(t) + 0.8Q'(t) + 64Q(t) = \cos(8t)$.

D. $Q''(t) + 17Q'(t) + 64Q(t) = 0$.

E. $Q''(t) + 64Q(t) = 0$.

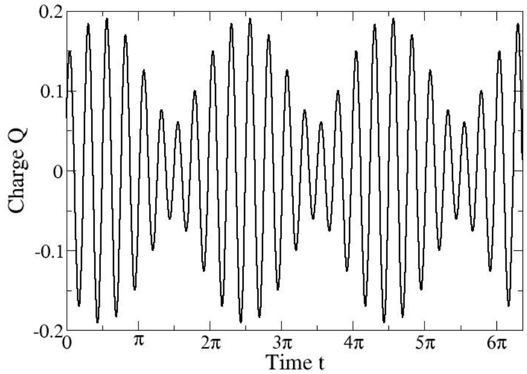
F. $Q''(t) + 64Q(t) = \cos(7t)$.

COMPLETE THIS TABLE:

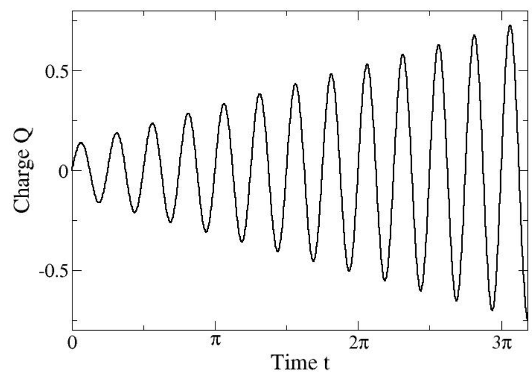
Equation	Graph
A	
B	
C	
D	
E	
F	

Hints:

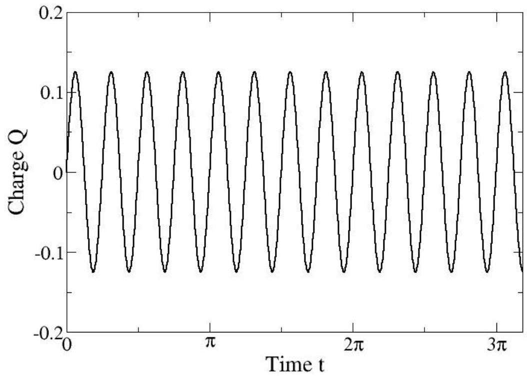
1. You don't need to solve each problem completely.
2. Look at the plots carefully and don't forget to check the axis scales.
3. No work need be shown.



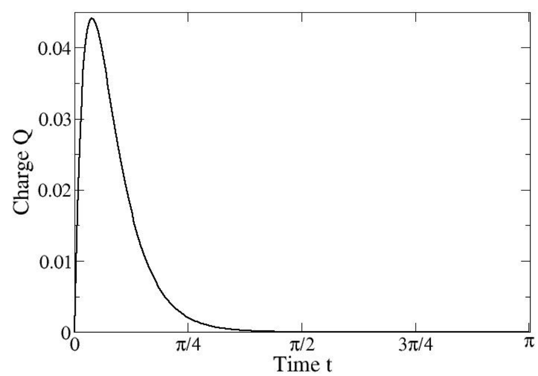
1.



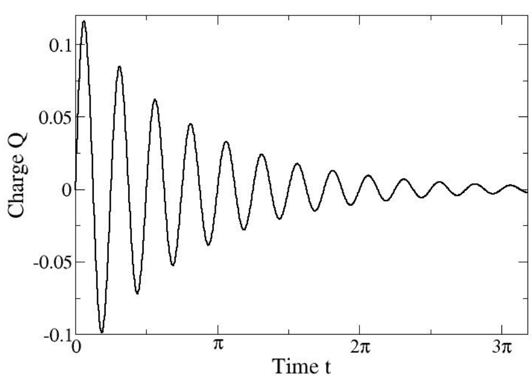
2.



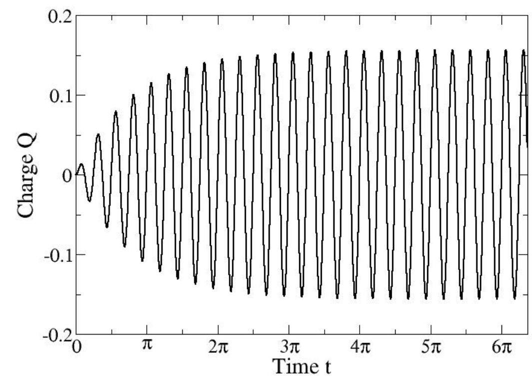
3.



4.



5.



6.

END OF EXAM

Table of Laplace Transforms

$f(t)$	$\mathcal{L}[f(t)] = F(s)$	
1	$\dfrac{1}{s}$	(1)
$e^{at}f(t)$	$F(s-a)$	(2)
$\mathcal{U}(t-a)$	$\dfrac{e^{-as}}{s}$	(3)
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$	(4)
$t^n f(t)$	$(-1)^n \dfrac{d^nF(s)}{ds^n}$	(5)
$f'(t)$	$sF(s) - f(0)$	(6)
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	(7)
$\int_0^t f(x)g(t-x)dx$	$F(s)G(s)$	(8)
$t^n \ (n = 0, 1, 2, \dots)$	$\dfrac{n!}{s^{n+1}}$	(9)
$\sin kt$	$\dfrac{k}{s^2+k^2}$	(10)
$\cos kt$	$\dfrac{s}{s^2+k^2}$	(11)
e^{at}	$\dfrac{1}{s-a}$	(12)

$f(t)$	$\mathcal{L}[f(t)] = F(s)$	
te^{at}	$\dfrac{1}{(s-a)^2}$	(13)
t^ne^{at}	$\dfrac{n!}{(s-a)^{n+1}}$	(14)
$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$	(15)
$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$	(16)
$e^{at}\sinh kt$	$\dfrac{k}{(s-a)^2-k^2}$	(17)
$e^{at}\cosh kt$	$\dfrac{s-a}{(s-a)^2-k^2}$	(18)
$t\sin kt$	$\dfrac{2ks}{(s^2+k^2)^2}$	(19)
$t\cos kt$	$\dfrac{s^2-k^2}{(s^2+k^2)^2}$	(20)
$t\sinh kt$	$\dfrac{2ks}{(s^2-k^2)^2}$	(21)
$t\cosh kt$	$\dfrac{s^2-k^2}{(s^2-k^2)^2}$	(22)

Trig identities

$$\begin{aligned} \sin(A \pm B) &= \sin A \cos B \pm \sin B \cos A \\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \end{aligned}$$