Be sure this exam has 14 pages including the cover

The University of British Columbia

MATH 215/255, Sections 101–105

Final Exam – December 2014

Family Name ______________________ **Given Name** ________________________

Student Number ___________________ **Signature** ___________________________

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No notes nor calculators.

Problem	Points	Score
1	8	
2	12	
3	10	
4	13	
5	8	
6	12	
7	10	
8	12	
9	15	
Total:	100	

1. Solve the initial value problem

$$(3xy + y^2) + (x^2 + xy)y' = 0, \quad y(1) = 2.$$

(3 points) (a) Verify that $\mu(x) = x$ is an integrating factor, that is, $x(3xy + y^2)dx + x(x^2 + xy)dy = 0$ is exact.

(5 points) (b) Solve the initial value problem.

2. A second order chemical reaction can be modeled by the equation

$$\frac{dy}{dx} = \alpha(y-p)(y-q),$$

where α, p and q are constants

(4 points) (a) Assume that $\alpha > 0$ and $p > q > 0$. Find equilibrium points and classify stabilities of these equilibrium points.

(4 points) (b) Assume that $\alpha = 1$, $p = 0$, $q = 1$ and $y(0) = -1$, solve the initial value problem and determine the limiting value of $y(x)$ as $x \to \infty$.

(4 points) (c) Assume that $\alpha = 1$, $p = q = 0$ and $y(0) = 1$. Use Euler's method to approximate $y(2)$ with the step size $h = 1$.

(5 points) 3. (a) Solve $y'' + 4y' + 13y = 0, \quad y(0) = 2, \quad y'(0) = 5.$

(5 points) (b) Find a particular solution of

$$y'' + 3y' = e^{-3t}$$

using **either** the method of undetermined coefficients, **or** the method of variation of parameters.

4. Consider a vibrating system described by the initial value problem

$$u'' + cu' + 4u = \cos 2t, \qquad u(0) = 0, \quad u'(0) = 2.$$

where $c > 0$ is the damping coefficient.

(5 points) (a) Find the steady periodic part of the solution (the part of the solution which remains as $t \to \infty$) of this problem, and find its amplitude. Do not find the transient part.

(2 points) (b) Let $A(c)$ denote the maximum amplitude of the steady state solutions of the systems

$$u'' + cu' + 4u = \cos \omega t, \qquad u(0) = 0, \quad u'(0) = 2$$

among all possible $\omega > 0$. What happens to $A(c)$ as $c \to 0_+$? Explain why.
Hint. You do not need to solve $A(c)$ explicitly.

(6 points) (c) Find a particular solution of

$$y'' + y = \frac{1}{\cos t}, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

Hint: $\int \tan t \, dt = -\ln|\cos t|, \quad \int \cot t \, dt = \ln|\sin t|$

(8 points) 5. Use the Laplace transform to solve the system $x''(t) + 2x'(t) + 2x(t) = 2$ with initial conditions $x(0) = 0$, $x'(0) = 0$.

(6 points) 6. (a) Solve the system $x''(t) + 4x(t) = \delta(t-2)$ with initial conditions $x(0) = 0$ and $x'(0) = 0$.

(6 points) (b) Find the Laplace transform of

$$f(t) = \begin{cases} 1 & \text{if } t < 1 \\ t & \text{if } 1 \leq t < 2 \\ 0 & \text{if } t \geq 2. \end{cases}$$

(7 points) 7. (a) Find general solution of

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad A = \begin{bmatrix} p & 4 \\ -1 & p \end{bmatrix},$$

assuming that p is real.

(3 points) (b) Describe the behaviour of the system (do not draw phase portrait) for all possible real values of p.

(12 points) 8. Solve

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} - \begin{bmatrix} 0 \\ 3t+2 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}.$$

9. Consider the nonlinear system

$$\frac{dx}{dt} = x^2 - \frac{y^2}{2} + 1, \qquad \frac{dy}{dt} = -4x - 2y. \tag{1}$$

(5 points) (a) Find the equilibria (critical points) for the system (1).

(6 points) (b) Find the Jacobian (partial derivative) matrix for the system (1), compute the linearized system at each equilibrium, and compute the eigenvalues for each of the coefficient matrices.

(4 points) (c) Identify the equilibrium of (1) at which the linearized system (not system (1)) has a center. Classify the other equilibrium and indicate whether it is (asymptotically) stable or unstable in system (1).

Table of Laplace transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}, \quad s > 0$
2. e^{-at}	$\frac{1}{s+a}, \quad s > -a$
3. t^n, n positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
4. $\sin(at)$	$\frac{a}{s^2+a^2}, \quad s > 0$
5. $\cos(at)$	$\frac{s}{s^2+a^2}, \quad s > 0$
6. $\sinh(at)$	$\frac{a}{s^2-a^2}, \quad s > \lvert a \rvert$
7. $\cosh(at)$	$\frac{s}{s^2-a^2}, \quad s > \lvert a \rvert$
8. $u(t-a)$	$\frac{e^{-as}}{s}, \quad s > 0$
9. $u(t-a)f(t-a)$	$e^{-as}F(s)$
10. $e^{-at}f(t)$	$F(s+a)$
11. $\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
12. $\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$
13. $\delta(t-a)$	e^{-as}
14. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - ... - f^{(n-1)}(0)$

Variation of parameters

If $y_1(x)$ and $y_2(x)$ are two solutions of $Ly = 0$, then the particular solution of $Ly = f(x)$ is

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x),$$

$$\begin{aligned} y_1u_1' + y_2u_2' &= 0, \\ y_1'u_1' + y_2'u_2' &= f(x). \end{aligned}$$