The University of British Columbia

Final Examination - April 23, 2013

Mathematics 215, Section 202

Closed book examination | Time: 2.5 hours

Last Name ____________ **First** ________ **Signature** ____________

Student Number ____________

Special Instructions:

No books, notes, or calculators are allowed.

Rules governing examinations

• Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.

• Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.

• No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.

• Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.

• Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
(a) speaking or communicating with other candidates, unless otherwise authorized;
(b) purposely exposing written papers to the view of other candidates or imaging devices;
(c) purposely viewing the written papers of other candidates;
(d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
(e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).

• Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.

• Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.

• Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1		10
2		10
3		8
4		22
5		12
6		12
7		12
8		14
Total		100

[10 points] **1.** Solve the initial value problem:

$$\frac{dy}{dx} = \frac{e^y}{x+2}, \qquad y(0) = 0, \qquad x \geq 0.$$

Also, determine the largest interval of positive x where this solution exists.

[10 points] **2.** Determine if the following equation is exact or not, and in either case, find the general solution:

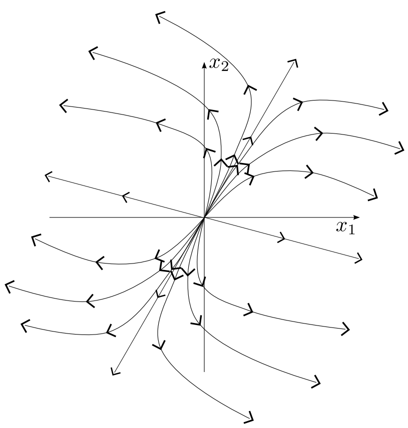
$$y dx + (x^2 y - x) dy = 0, \qquad x > 0.$$

[8 points] **3.** Let $-\infty < y_0 < \infty$ and consider the following initial value problem:

$$\frac{dy}{dt} = e^y - e^{-y}, \qquad y(0) = y_0, \qquad t \geq 0.$$

(a) Find all equilibrium solution(s) of the differential equation above.
(b) Determine all values of y_0 such that $y(t)$ is **increasing** for all $t \geq 0$ where the solution to the initial value problem exists.
(c) Classify the equilibrium solution(s) as asymptotically stable or unstable. Provide a brief reason.

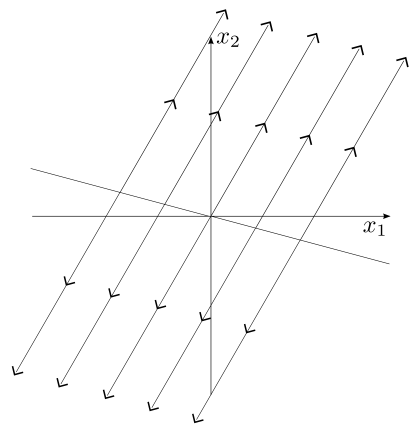
[3 points each] **4** (a). Each of the following four pictures are phase portraits of the system of equations $\boldsymbol{x}' = A\boldsymbol{x}$, where A is a constant, 2×2 matrix. For each picture, **check the appropriate boxes** to indicate the properties of the eigenvalues of A.
In the blank, indicate the sign (**positive**, **negative**, or **zero**) of **each** eigenvalue (or of its real part).



Real: ☐ Non-real: ☐

Repeated: ☐ Non-repeated: ☐

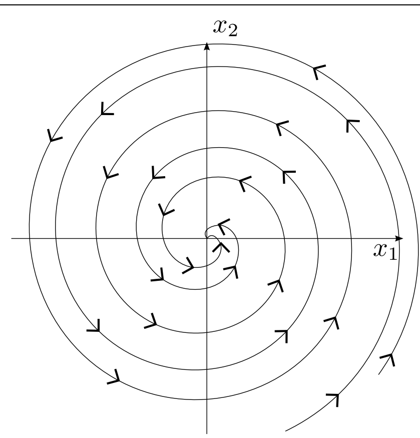
Sign(s) of eigenvalue(s):



Real: ☐ Non-real: ☐

Repeated: ☐ Non-repeated: ☐

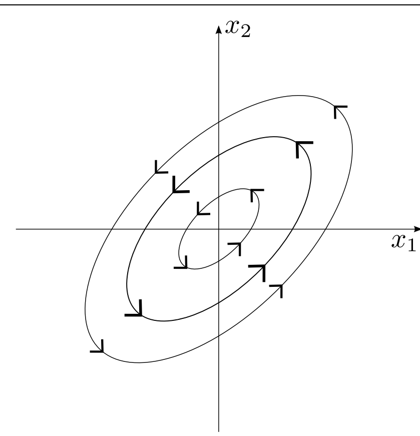
Sign(s) of eigenvalue(s):



Real: ☐ Non-real: ☐

Repeated: ☐ Non-repeated: ☐

Sign(s) of eigenvalue(s):

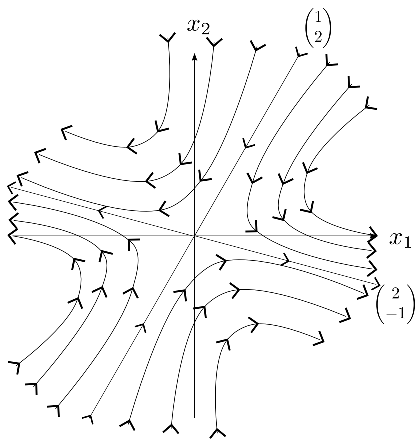


Real: ☐ Non-real: ☐

Repeated: ☐ Non-repeated: ☐

Sign(s) of eigenvalue(s):

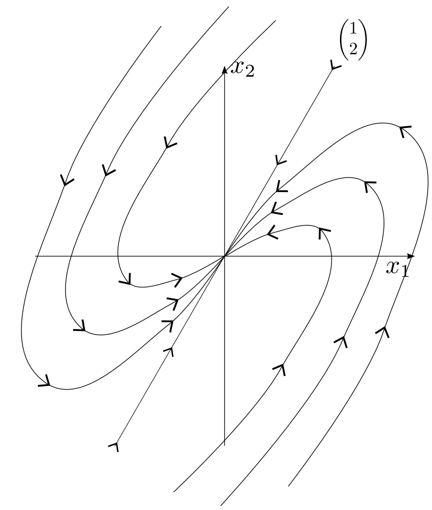
[10 points] (b). For the two pictures below, **check the appropriate boxes** to indicate the properties of the eigenvalues of A. Also, write the sign (**positive**, **negative**, or **zero**) of all eigenvalue(s) or their real part(s), and indicate **which eigenvectors** (labelled in the diagrams) are associated to which eigenvalues.



Real: ☐ Non-real: ☐

Repeated: ☐ Non-repeated: ☐

Sign(s) of eigenvalue(s) and their corresponding eigenvector(s):



Real: ☐ Non-real: ☐

Repeated: ☐ Non-repeated: ☐

Sign(s) of eigenvalue(s) and their corresponding eigenvector(s):

[12 points] **5.** Use the method of diagonalization to transform the following system of equations into a **decoupled** one:

$$\boldsymbol{x}' = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} t \\ e^{t^2} \end{pmatrix}$$

and **you must explicitly calculate all matrices involved**.

Here $\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$. **You do not need to solve the system.**

[12 points] **6.** For the following system:

$$\begin{aligned} x' &= -2x + x^2 + xy \\ y' &= -3y + 2y^2 + xy \end{aligned}$$

find all critical points and classify them according to their types and stability.

[12 points] **7.** Suppose that $p(t)$ and $q(t)$ are continuous functions on $(0,\infty)$. Find a particular solution to the inhomogeneous equation

$$ty'' + p(t)y' + q(t)y = t^3, \qquad t > 0,$$

given that $\{t,\ t^2\}$ is a fundamental set of solutions to the homogeneous equation

$$ty'' + p(t)y' + q(t)y = 0.$$

[14 points] **8.** Use the Laplace transform to solve the initial value problem:

$$y'' + 2y = u_\pi(t)\sin t + u_{2\pi}(t)\sin t, \quad y(0) = y'(0) = 1,$$

where $u_\pi(t)$, $u_{2\pi}(t)$ are the unit step functions with steps at π and 2π, respectively.

THE END

Table of Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
$\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
$\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \|a\|$
$\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \|a\|$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
$t^n e^{at}, \quad n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$e^{ct}f(t)$	$F(s-c)$
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$
$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
$\delta(t-c)$	e^{-cs}
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
$(-t)^n f(t)$	$F^{(n)}(s)$