Be sure this exam has 13 pages including the cover

The University of British Columbia

MATH 215, Section 201

Final Exam – April 2010

Family Name ______________________ **Given Name** ________________________

Student Number ___________________ **Signature** __________________________

No notes nor calculators.

Rules Governing Formal Examinations:

1. Each candidate must be prepared to produce, upon request, a UBCcard for identification;

2. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions;

3. No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination;

4. Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action;
(a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
(b) Speaking or communicating with other candidates;
(c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received;

5. Candidates must not destroy or mutilate any examination material; must hand in all examination papers, and must not take any examination material from the examination room without permission of the invigilator; and

6. Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

problem	max	score
1.	10	
2.	10	
3.	18	
4.	10	
5.	20	
6.	10	
7.	10	
8.	12	
total	100	

(7 points) 1. (a) Find the solution of the given initial value problem in explicit form:

$$\frac{d}{dt}y = \frac{y^2}{t}, \quad y(1) = 2.$$

(3 points) (b) Determine the interval of existence of the solution in (a).

(10 points) 2. Solve the initial value problem

$$\frac{dy}{dx} = -\frac{2x + y^2 + 1}{y}, \quad y(0) = 1.$$

(6 points) 3. (a) Solve $y'' - 5y' - 6y = 0, \quad y(0) = 2, \quad y'(0) = -9.$

(6 points) (b) Use the method of reduction of order to find a second solution $y_2(t)$ of the equation

$$t^2y'' + ty' - 4y = 0, \quad t > 0; \quad y_1(t) = t^2$$

which is independent of y_1.

(6 points) (c) Find the general solution of

$$y'' + 4y' + 4y = t^{-2}e^{-2t},$$

given that $y_1(t) = e^{-2t}$ and $y_2(t) = te^{-2t}$ solve $y'' + 4y' + 4y = 0$.

(10 points) 4. Suppose the motion of a certain mass-spring system satisfies the differential equation

$$u'' + \gamma u' + 4u = 0, \quad u(0) = 1, \quad u'(0) = a, \tag{1}$$

with units m, kg, and s. Here $u(t)$ is the displacement from the equilibrium position, $\gamma > 0$ is the damping coefficient, and a is a real parameter.

(a) Determine the range of γ so that the solution $u(t)$ is not oscillatory.

(b) Suppose $\gamma = 5$. Determine the range of real a so that the solution of (1) returns to the equilibrium point $u = 0$ at some $t > 0$.

(6 points) 5. (a) Find the Laplace transform $Y(s)$ of the solution of

$$y'' + y = g(t), \quad g(t) = \begin{cases} 1 & \text{if } 0 \le t < 2 \\ \cos(t-2) & \text{if } t \ge 2 \end{cases}$$
$$y(0) = 1, \qquad y'(0) = 2.$$

Do NOT attempt to find $y(t)$!

(5 points) (b) Find the inverse Laplace transform of $F(s) = \dfrac{s}{(s+1)(s+2)^2}$.

(9 points) (c) Find the solution to the initial value problem

$$y'' + 2y' + 2y = \delta(t - \pi); \quad y(0) = 1, \quad y'(0) = -2.$$

Here $\delta(t)$ represents the Dirac delta function.

(10 points) 6. For the linear system

$$\frac{d}{dt}\mathbf{x} = A\mathbf{x}, \quad A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix},$$

find the general solution and the matrix e^{tA}.

(10 points) 7. Solve

$$\frac{d}{dt}\mathbf{x} = A\mathbf{x} - \begin{bmatrix} e^t \\ e^t \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix},$$

given that the general solution of $\frac{d}{dt}\mathbf{x} = A\mathbf{x}$ is $\mathbf{x}_c(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t}$.

8. Consider the nonlinear system

$$\frac{dx}{dt} = 1 - y, \qquad \frac{dy}{dt} = x^2 - y^2. \tag{2}$$

(4 points) (a) Find the critical points for the system (2).

(2 points) (b) Find the Jacobian (partial derivative) matrix for the system (2).

(6 points) (c) Use the linear systems near each critical point to classify them, specifying both the type of critical point and whether it is (asymptotically) stable or unstable.

Table of Laplace transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}, \quad s > 0$
2. e^{at}	$\frac{1}{s-a}, \quad s > a$
3. t^n, n positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
4. t^p, $p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5. $\sin(at)$	$\frac{a}{s^2+a^2}, \quad s > 0$
6. $\cos(at)$	$\frac{s}{s^2+a^2}, \quad s > 0$
7. $\sinh(at)$	$\frac{a}{s^2-a^2}, \quad s > \lvert a \rvert$
8. $\cosh(at)$	$\frac{s}{s^2-a^2}, \quad s > \lvert a \rvert$
9. $e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
10. $e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
11. $t^n e^{at}$, n positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
12. $u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$
14. $e^{ct}f(t)$	$F(s-c)$
15. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$
16. $\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
17. $\delta(t-c)$	e^{-cs}
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - ... - f^{(n-1)}(0)$
19. $(-t)^n f(t)$	$F^{(n)}(s) = (\frac{d}{ds})^n F(s)$