Be sure that this examination has three pages.

The University of British Columbia
Final Examinations – December 2006

Mathematics 215
Elementary Differential Equations I

Mathematics 255
Ordinary Differential Equations

Time: $2\frac{1}{2}$ hours

Special instructions:

(1) One $8\frac{1}{2}'' \times 11''$ page of notes may be used, but no other aids are permitted. In particular, *calculators and cell phones are not allowed.*

(2) Answers must be justified to receive full credit.

(3) A table of Laplace transforms is attached.

Marks

[15] 1. (a) Find the general solution of the equation

$$\frac{dy}{dt} = \frac{t^2 - y}{t}.$$

(b) Find all values of the constants a and b so that the differential equation

$$\frac{dy}{dx} = -\frac{y^a}{2xy + 2bxy^2}$$

is exact, then for these values of a and b find the solution of the equation that also satisfies the initial condition $y(1) = 1$.

[20] 2. Consider the autonomous equation

$$\frac{dy}{dt} = (y-1)(y-5), \qquad -\infty < y < +\infty.$$

(a) Find all critical points (equilibria) and sketch the one-dimensional phase portrait (i.e. phase line) for the equation.

(b) If $y_1(t)$ and $y_2(t)$ are two solutions of the equation which satisfy the initial conditions $y_1(0) = 4$ and $y_2(0) = 3$, find $\lim_{t\to\infty} |y_1(t) - y_2(t)|$ if it exists.

(c) Plot the direction field of the equation, and on the same plot sketch the graph of the solution $y_1(t)$ versus t, if $y_1(t)$ solves the initial value problem with initial condition $y_1(0) = 4$. You do not need to find the solution $y_1(t)$ explicitly, but indicate clearly where the solution is increasing or decreasing, and where its graph is concave up or concave down.

(d) Use Euler's method (i.e. tangent line method or explicit Euler's method) with step size $h = 1$ to find an approximation to the solution $y_1(t)$ at $t = 2$, if $y_1(0) = 4$. Sketch the approximate solution and the exact solution on the same graph.

[25] 3. (a) Given that $y_1(t) = t$ is a solution (you do not need to verify this) of

$$t^2 y'' - ty' + y = 0, \quad t > 0,$$

find another linearly independent solution $y_2(t)$, and verify that $y_1(t)$ and $y_2(t)$ are linearly independent solutions on $t > 0$.

(b) Find the general solution of

$$y'' + 9y = \cos(3t).$$

(c) A mass of $m = 1$ kilogram, hanging from a spring with spring constant $k =$ 9 Newtons/metre, experiences no friction and is acted on by an external force $\cos(\omega t)$ Newtons. For what value of ω does resonance occur? *Briefly* explain what resonance is.

[10] 4. Solve the initial value problem

$$y'' + y = \delta(t - \pi), \qquad y(0) = 0, \quad y'(0) = 0.$$

[15] 5. Find the general solution of the linear system

$$\mathbf{x}' = \begin{pmatrix} 0 & 6 \\ 2 & 4 \end{pmatrix} \mathbf{x}.$$

[15] 6. Consider the system of nonlinear equations

$$\begin{aligned} \frac{dx}{dt} &= 3 - 3y^2, \\ \frac{dy}{dt} &= 2x - 2y^2. \end{aligned}$$

(a) Find all critical points (equilibria) of the system.

(b) For each critical point, classify its type (node, saddle point, spiral point, or centre) and determine its stability (asymptotically stable, stable but not asymptotically stable, or unstable).

(c) Draw the phase portrait of the system.

Total marks [100]

The End