The University of British Columbia

MATH 200 Final Examination -Dec 12, 2016

Closed book examination

Time: 150 minutes

Special Instructions:

No memory aids, calculators, or electronic devices of any kind are allowed on the test. Where blanks are provided for answers, put your final answers in them. UNLESS OTHERWISE SPECIFIED, SHOW ALL YOUR WORK; little or no credit will be given for answers without the correct accompanying work. Numerical answers should be left in calculator-ready form, unless otherwise indicated. If you need more space than the space provided, use the back of the previous page.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.

  (b) Speaking or communicating with other candidates.

  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

1		9
2		10
3		11
4		9
5		13
6		7
7		10
8		11
Total		80

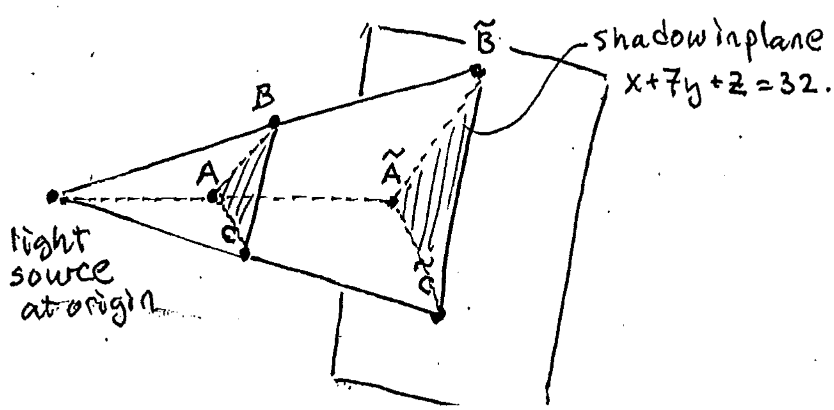
1. Let $A = (0, 2, 2), B = (2, 2, 2), C = (5, 2, 1)$

3pts (a) The line which contains A and is perpendicular to the triangle ABC has parametric equations:

$$\begin{cases} x = \\ y = \\ z = \end{cases}$$

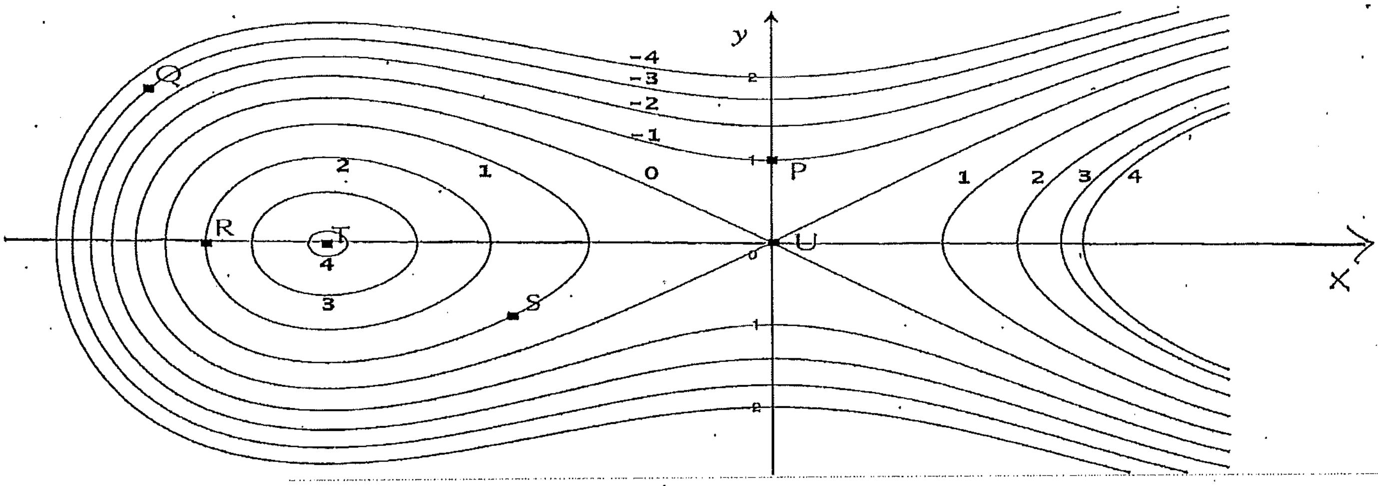
3pts (b) The set of all points P such that $\overrightarrow{PA}$ is perpendicular to $\overrightarrow{PB}$ form a Plane/ Line/ Sphere/ Cone/ Paraboloid/ Hyperboloid (circle one) in space which satisfies the equation: ________________________,

3pts (c) If a light source at the origin shines on triangle ABC making a shadow on the plane $x + 7y + z = 32$ (see diagram), then $\tilde{A} = ($____, ____, ____$)$.



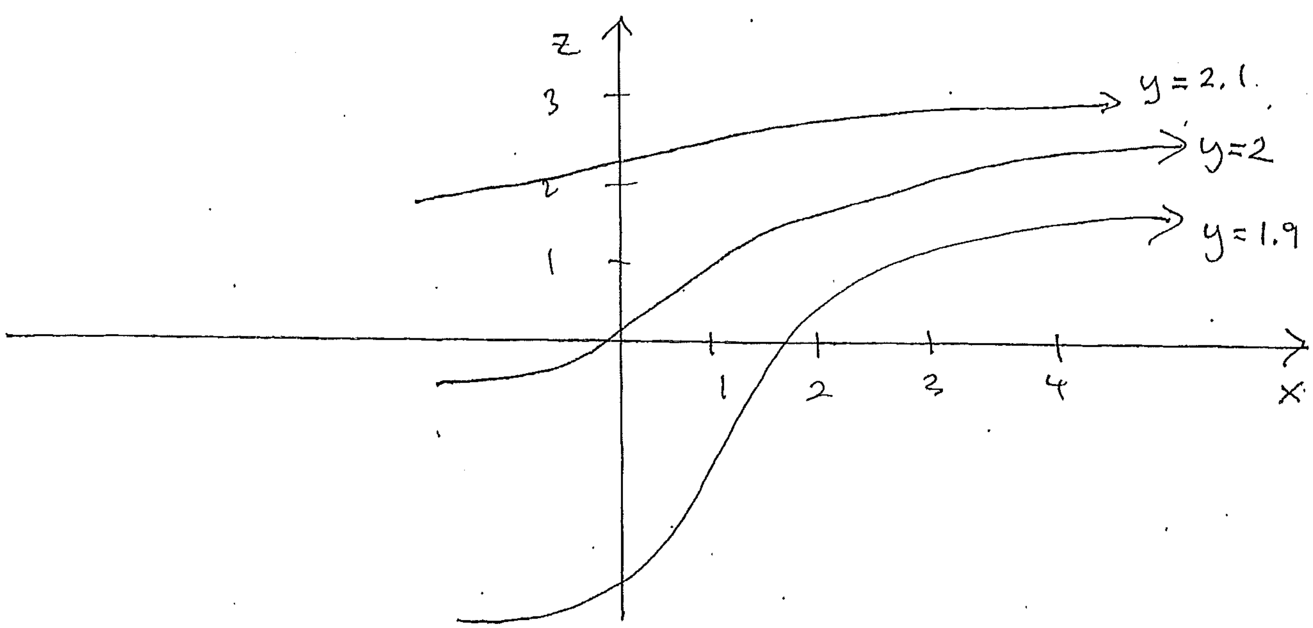
2a) Some level curves of a function f(x, y) are plotted in the xy plane below. For each of the four statements below, circle the letters of **all** points in the diagram where the situation applies. For example, if the statement were "These points are on the y-axis:" you would circle both P and U, but none of the other letters. You may assume that a local maximum occurs at Point T.

i) gradient f is zero	P R S T U
ii) f has a saddle point	P R S T U
iii) the partial derivative f_y is positive	Q R S T U
iv) the directional derivative of f in the direction <0,-1> is negative	Q R S T U



2b) The diagram below shows three "y traces" of a graph z=F(x, y) plotted on xz axes. (namely, the intersections of the surface z=F(x, y) with the three planes (y=1.9, y=2, y=2.1). For each statement below, circle the correct word.

i) The first order partial derivative F_x (1,2) is	positive/negative/zero (circle one)
ii) F has a critical point at (2,2)	true/false (circle one)
iii) The second orderpartial derivative F_xy (1,2) is	positive/negative/zero (circle one)



3. Consider the functions $F(x, y, z) = z^3 + xy^2 + xz$ and $G(x, y, z) = 3x - y + 4z$. You are standing at the point $P(0, 1, 2)$.

5pts (a) If you jump from P to $Q(0.1, 0.9, 1.8)$ then the amount by which F changes is approximately:

______________________________ (use linear approximation).

3pts (b) If you jump from P in the direction along which G increases most rapidly, then F will increase/decrease (circle one and explain below).

3pts (c) You jump from P in a direction $\langle a, b, c\rangle$ along which rate of change of F and G are both zero. An example of such a direction is $\langle a, b, c\rangle =$ ____________ (need not be unit vector).

4. Suppose $f(x, y)$ is twice differentiable (with $f_{xy} = f_{yx}$), and $x = r\cos\theta$ and $y = r\sin\theta$.

7pts (a) Fill blanks below in terms of functions depending on r and/or θ, and partial derivatives of f with respect to x, y.

$f_\theta =$ ______________________________

$f_r =$ ______________________________

$f_{r\theta} =$ ________ $f_x +$ ________ $f_y +$ ________ $f_{xx} +$ ________ $f_{xy} +$ ________ f_{yy}

2pts (b) Let $g(x, y)$ be another function satisfying $g_x = f_y; g_y = -f_x$. Fill blanks below with constants or functions depending on r and/or θ

$f_r =$ __________ g_θ

$f_\theta =$ __________ g_r

5. The temperature in the plane is given by $T(x, y) = e^y(x^2 + y^2)$.

(a) 3pts (i) To find the warmest and coolest points on the **circle** $x^2 + y^2 = 100$ using Lagrange multipliers, we must solve the following system:

$$\left\{ \begin{array}{l} \\ \\ \\ \end{array} \right.$$

4pts (ii) By solving the above system, we conclude the warmest point on the **circle** is: ________________, an the coolest point is: ________________

(b) Take the same termperature function as in part (a) ($T(x, y) = e^y(x^2 + y^2)$.)

1pts (i) To find the critical points of $T(x, y)$ we must solve the following system:

$$\begin{cases} \\ \\ \end{cases}$$

3pts (ii) By solving the above system we conclude the critical points are: ____________________

2pts (c) The coolest point on the solid disc $x^2 + y^2 \leq 100$ is ________________.

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6. Let $I = \int_0^{\sqrt{\pi}/2} \int_{x^2}^{1} x^3 \sin(y^3)\,dy\,dx$

2pts (a) Sketch the corresponding region of integration in the xy plane (label your sketch sufficeintly so that one could conversely use it to determine the limits of double integration)

5pts (a) Evaluate I.

7. Let S be the region in the first octant (so $x, y, z \geq 0$) which lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $(z-1)^2 + x^2 + y^2 = 1$. Let V be its volume.

3pts (a) Fill in the blanks below using cylindrical coordinates (no explanation required)

$$V = \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} (\_\_\_\_\_\_\_\_\_\_\_\_) d\_ d\_ d\_$$

3pts (b) Fill in the blanks below using spherical coordinates (no explanation required)

$$V = \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} (\_\_\_\_\_\_\_\_\_\_\_\_) d\_ d\_ d\_$$

4pts (c) Calculate V using either (not both) of the integrals above.

8. [11pts] Let E be the region bounded by the planes $y = 0, y = 2, y + z = 3$ and the surface $z = x^2$. Consider the integral

$$I = \int\int\int_E f(x, y, z)dV.$$

Fill in the blanks below (No explanations required. Also, in each part below, you may only need one integral to express your answer, in which case leave the other blank)

(a) $I = \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} f(x, y, z)dzdxdy + \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} f(x, y, z)dzdxdy$

b) $I = \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} f(x, y, z)dxdydz + \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} f(x, y, z)dxdydz$

c) $I = \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} f(x, y, z)dydxdz + \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} f(x, y, z)dydxdz$

The End

FORMULA SHEET

$$\langle a_1, a_2, a_3\rangle \cdot \langle b_1, b_2, b_3\rangle = a_1b_1 + a_2b_2 + a_3b_3 = |\mathbf{a}|\,|\mathbf{b}|\,\cos\theta$$

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a}$$

$$\text{Det}(\mathbf{a},\mathbf{b}) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 = -\begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix} = -\text{Det}(\mathbf{b},\mathbf{a})$$

$$\text{Det}(\mathbf{a},\mathbf{b},\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\text{Det}(\mathbf{a},\mathbf{b},\mathbf{c}) = \mathbf{a}\cdot\left\langle \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}, -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}, \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \right\rangle = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}).$$

$$\mathbf{b}\times\mathbf{c} = \left\langle \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}, -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}, \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \right\rangle$$

$$|\mathbf{b}\times\mathbf{c}| = |\mathbf{b}|\,|\mathbf{c}|\,|\sin\theta|$$

$$\text{Det}(\mathbf{a},\mathbf{b},\mathbf{c}) = -\text{Det}(\mathbf{b},\mathbf{a},\mathbf{c}) \quad \text{so} \quad \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = -\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c})$$

$$x = x_0 + at,\; y = y_0 + bt,\; z = z_0 + ct, \qquad \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

$$\mathbf{r}_0 + t\mathbf{v}, \quad (1-t)\mathbf{r}_0 + t\mathbf{r}_1$$

$$\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}_0) = 0, \quad a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$\text{dist}(ax+by+cz+d_1 = 0, ax+by+cz+d_2 = 0) = \frac{|d_1-d_2|}{|\langle a,b,c\rangle|}$$

Distance from a point $P(x_1, y_1, z_1)$ to the plane $ax+by+cz+d=0$ is

$$D = |ax_1 + by_1 + cz_1 + d| \;/\; \sqrt{a^2+b^2+c^2}.$$

$$\begin{aligned} 1/2 &= \sin(30^\circ) = \sin(\pi/6) = \cos(60^\circ) = \cos(\pi/3) \\ 1/\sqrt{2} &= \sin(45^\circ) = \sin(\pi/4) = \cos(45^\circ) = \cos(\pi/4) \\ \sqrt{3}/2 &= \cos(30^\circ) = \cos(\pi/6) = \sin(60^\circ) = \sin(\pi/3) \end{aligned}$$

$$\cos(2\alpha) = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1$$

$$\begin{aligned} L(x,y) = f(a,b) + {} & f_x(a,b)(x-a) + && f_y(a,b)(y-b) \\ f(x,y) - f(x_0,y_0) = z - z_0 = {} & f_x(x_0,y_0)(x-x_0) && +f_y(x_0,y_0)(y-y_0) \\ \Delta f = \Delta z = {} & f_x(x_0,y_0)\Delta x && +f_y(x_0,y_0)\Delta y \\ df = dz = {} & f_x(x_0,y_0)\,dx && +f_y(x_0,y_0)\,dy \end{aligned}$$

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

Directional derivative ($\mathbf{u}$ a unit vector):

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \langle f_x, f_y, f_z\rangle \cdot \mathbf{u}$$

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$$

$$f_{xx}, \quad D = f_{xx}f_{yy} - (f_{xy})^2$$

Local min: $f_{xx} > 0$, $D > 0$; local max: $f_{xx} < 0$, $D > 0$; saddle: $D < 0$; degenerate/indeterminate: $D = 0$.

Lagrange multipliers to maximize/minimize $f = f(x,y)$ or $f = f(x,y,z)$ subject to $g = C =$ Constant where $g = g(x,y)$ or $g = g(x,y,z)$:

$$\nabla f = \lambda \nabla g(, \quad g = C).$$

Mass and centre of mass, density $\rho = \rho(x,y)$:

$$m = \int\int_D \rho(x,y)\,dA, \quad \overline{x} = (1/m)\int\int_D x\rho(x,y)\,dA, \quad \overline{y} = (1/m)\int\int_D y\rho(x,y)\,dA$$

Polar/cylindrical coordinates: $x = r\cos\theta$, $y = r\sin\theta$,

$$dA = dx\,dy = r\,dr\,d\theta, \quad dV = dx\,dy\,dz = r\,dz\,dr\,d\theta$$

Spherical (like polar in x, y with radius $r = \rho\sin\phi$, $z = \rho\cos\phi$, so ϕ measures angle with positive z-axis, $0 \le \phi \le \pi = 180^\circ$):

$$x = (\rho\sin\phi)\cos\theta, \quad y = (\rho\sin\phi)\sin\theta, \quad z = \rho\cos\phi, \quad dV = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$