The University of British Columbia

Final Examination - April 24, 2014

Mathematics 200

Circle one: Section 201 MWF 9-10 Section 202 MWF 11-12

Closed book examination Time: 2.5 hours

Last Name ____________ **First** ____________ **Signature** ____________

Student Number ____________

Special Instructions:

No notes or calculators are allowed. Answer all questions on the sheets provided. Use the backs of the sheets and blank sheets if necessary. Show your final answer clearly by drawing a box around it.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.

  (b) Speaking or communicating with other candidates.

  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		18
2		17
3		10
4		10
5		10
6		10
7		10
8		15
Total		100

Formulas

- Projection of $\vec{v}$ onto $\vec{u}$:

$$proj_{\vec{u}}\vec{v} = \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\vec{u}.$$

- Cross product

$$\langle a, b, c\rangle \times \langle d, e, f\rangle = \langle bf - ce, -(af - cd), ae - bd\rangle.$$

- Equations of the line through (x_0, y_0, z_0) in the direction $\vec{v} = \langle a, b, c\rangle$:

$$(x, y, z) = (x_0 + ta, y_0 + tb, z_0 + tc), \qquad \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

- Tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0):

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

- Tangent plane to the level surface $F(x, y, z) = k$ at the point (x_0, y_0, z_0):

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

- Linear approximation of $f(x, y)$ at the point (x_0, y_0):

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

- differential

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

- Chain rule:

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

- Implicit differentiation if $y(x)$ is given by $F(x, y) = 0$:

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

- Directional derivative:

$$D_{\vec{u}}f = \nabla f \bullet \vec{u} = \langle f_x, f_y\rangle \bullet \vec{u}.$$

- Classification of critical points of $f(x,y)$. Set $D = f_{xx}f_{yy} - f_{xy}^2$. Then
  - if $D > 0$ and $f_{xx} > 0$, then local minimum;
  - if $D > 0$ and $f_{xx} < 0$, then local maximum;
  - if $D < 0$, then saddle.
- Lagrange multiplier equations to find min/max of f with constraint $g = k$:

$$\nabla f = \lambda \nabla g, \qquad g = k.$$

- Iterated integrals:

$$\iint_D f dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx.$$

- Mass and center of mass, given density $\rho(x,y,z)$:

$$m = \iiint_E \rho(x,y,z)dV, \quad M_x = \iiint_E x\rho(x,y,z)dV, \quad \bar{x} = M_x/m, \quad \ldots.$$

- Cylindrical coordinates

$$x = r\cos\theta, y = r\sin\theta, z = z; \quad \iiint_E f(x,y,z)dV = \iiint_B f(r\cos\theta, r\sin\theta, z)rdzdrd\theta.$$

- Spherical coordinates

$$x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi; \qquad \iiint_E f dV = \iiint_B f\rho^2 \sin\phi d\rho d\phi d\theta.$$

- Sin and Cos:

θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\sin\theta$	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$
$\cos\theta$	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$

PROBLEM 1. Consider two planes W_1, W_2, and a line M defined by:

$$W_1 : \ -2x + y + z = 7, \quad W_2 : \ -x + 3y + 3z = 6, \quad M : \ \frac{x}{2} = \frac{2y - 4}{4} = z + 5.$$

a. Find a parametric equation of the line of intersection L of W_1 and W_2.

b. Find the distance from L to M.

Continue on the next page.

c. Find the area of the parallelogram on W_2 $(-x + 3y + 3z = 6)$ defined by $0 \leq x \leq 3$, $0 \leq y \leq 2$.

PROBLEM 2. Let the pressure P and temperature T at a point (x, y, z) be

$$P(x, y, z) = \frac{x^2 + 2y^2}{1 + z^2}, \qquad T(x, y, z) = 5 + xy - z^2,$$

a. If the position of an airplane at time t is

$$(x(t), y(t), z(t)) = (2t, t^2 - 1, \cos t),$$

find $\frac{d}{dt}(PT)^2$ at time $t = 0$ as observed from the airplane.

Continue on the next page.

$$P(x,y,z) = \frac{x^2 + 2y^2}{1 + z^2}, \qquad T(x,y,z) = 5 + xy - z^2,$$

b. In which direction should a bird at the point $(0, -1, 1)$ fly if it wants to keep both P and T constant. (Give one possible direction vector. It does not need to be a unit vector.)

Continue on the next page.

c. An ant crawls on the surface $z^3+zx+y^2=2$. When the ant is at the point $(0,-1,1)$, in which direction should it go for maximum increase of the temperature $T=5+xy-z^2$? Your answer should be a vector $\langle a,b,c\rangle$, not necessarily of unit length. (Note that the ant cannot crawl in the direction of the gradient because that leads off the surface. The direction vector $\langle a,b,c\rangle$ has to be on the tangent plane to the surface.)

PROBLEM 3. Consider the function

$$f(x, y) = 3kx^2y + y^3 - 3x^2 - 3y^2 + 4,$$

where $k > 0$ is a constant. Find and classify all critical points of $f(x, y)$ as local minima, local maxima, saddle points or points of indeterminate type. Carefully distinguish the cases $k < \frac{1}{2}$, $k = \frac{1}{2}$ and $k > \frac{1}{2}$.

Continue on the next page.

Problem 4. Find the largest and smallest values of

$$f(x, y, z) = 6x + y^2 + xz$$

on the sphere $x^2 + y^2 + z^2 = 36$. Determine all points at which these values occur.

PROBLEM 5. Let D be the region in the xy-plane bounded on the left by the line $x = 2$ and on the right by the circle $x^2 + y^2 = 16$. Evaluate

$$\iint_D (x^2 + y^2)^{-3/2} dA.$$

PROBLEM 6. **a.** Let

$$I = \int_0^2 \int_0^x f(x,y)dydx + \int_2^6 \int_0^{\sqrt{6-x}} f(x,y)dydx.$$

Express I as an integral where we integrate first with respect to x.

b. Let

$$J = \int_0^1 \int_0^x \int_0^y f(x,y,z)dzdydx.$$

Express J as an integral where the integrations are to be performed in the order x first, then y, then z.

PROBLEM 7. Let E be the solid lying above the surface $z = y^2$ and below the surface $z = 4 - x^2$. Evaluate

$$\iiint_E y^2 dV.$$

Hint: you may need to use the half angle formulas:

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}, \quad \cos^2\theta = \frac{1+\cos 2\theta}{2}.$$

Problem 8. Let E be the solid

$$0 \le z \le \sqrt{x^2+y^2}, \quad x^2+y^2 \le 1,$$

and consider the integral

$$I = \iiint_E z\sqrt{x^2+y^2+z^2}\,dV.$$

a. Write the integral I in cylindrical coordinates.

b. Write the integral I in spherical coordinates.

Continue on the next page.

c. Evaluate the integral I using either form.

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