The University of British Columbia

Final Examination - April 20, 2016

Mathematics 152

All Sections

Closed book examination. No calculators. Time: 2.5 hours

Last Name __________________ **First** ____________ **Signature** _________________

Section : ____________________

Student Number ____________________

Instructor : ____________________

Special Instructions:

No books, notes, calculators or any electronic devices are allowed. Show all your work. In part B questions, little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practises shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

part A		30
B1		5
B2		5
B3		5
B4		5
B5		5
B6		5
Total		60

Part A - Short Answer Questions, 1 mark each

A1: Calculate the projection of the vector $[3, 1, 5]$ onto the vector $[2, 2, 4]$.

A2: Find the area of the triangle ABC in the plane, where $A = (3, 2)$, $B = (1, 3)$ and $C = (2, 5)$.

A3: Find the eigenvalues of the 2×2 matrix below. It is *not* necessary to find the corresponding eigenvectors.

$$\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$$

A4: Let A be a 5×5 matrix with $\det(A) = 10$. What is $\det(A^{-1})$?

Questions A5-A6 below involve the vectors

$$\mathbf{u} = [1, 1, a] \quad \text{and} \quad \mathbf{v} = [1, 2, 3]$$

For each question A5-A6 below justify your answer with a short computation or a short justification in words. Note that the vector $\mathbf{u}$ has a constant a in the last component.

A5: For what value or values of a (if any) are $\mathbf{u}$ and $\mathbf{v}$ perpendicular?

A6: For what value or values of a (if any) are $\mathbf{u}$ and $\mathbf{v}$ parallel?

A7: What is the result of the following MATLAB commands?

```
A = [1 2 3 4; 1 1 1 1; 9 8 7 6];
A(:,3)
```

A8: Find all solutions (x, y, z) to the linear system

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 5 \\ 2x & + & 2y & & & = & 6 \\ & & 2y & + & 4z & = & 8 \end{array} .$$

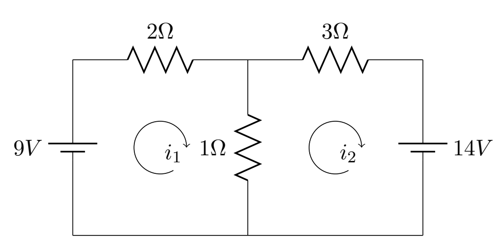
For questions A9 and A10 below, consider the homogeneous system of equations represented by this augmented matrix in reduced row echelon form:

$$\left[\begin{array}{ccccc|c} 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

A9: What is the rank of the augmented matrix above?

A10: Write a parametric form for all solutions to the system above.

A11: Solve for the loop currents i_1 and i_2 in the circuit to the right.



A12: Find the distance from the point (1, 2) to the line $x + y = 0$ in the plane.

A13: Calculate the determinant of this matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 10 & 0 & 3 & 10 \\ 7 & 1 & 5 & 9 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

A14: Find the area of the parallelogram with vertices at $(2, -2)$, $(3, 1)$, $(5, 6)$, and $(4, 3)$.

A15: Consider the line L passing through the point $P = [3, 2, 2]$ and which is perpendicular to the plane containing the points $A = [1, 0, 1]$, $B = [0, 1, 1]$, and $C = [-1, 0, 1]$. Give a parametric equation for L.

A16: Consider the matrix representation A of a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$. Circle *all* correct answers below:

(a) A is invertible.

(b) A has three rows.

(c) A has three columns.

(d) $T(\mathbf{x}) = \mathbf{0}$ for some $\mathbf{x} \neq \mathbf{0}$.

(e) $A = A^T$.

A17: Consider a linear system with 7 equations for 8 unknowns. Circle *all* possible types of solution sets that could result:

(a) The system has no solutions.

(b) The system has a unique solution.

(c) The system has exactly 8 distinct solutions.

(d) The system has a one-parameter family of solutions.

(e) The system has a two-parameter family of solutions.

A18: Find a constant a so that the following set of vectors is linearly *dependent*:

$$\{[a, 0, 1], [1, 2, 1], [4, 1, 3]\}.$$

A19: Find the matrix of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(\mathbf{v}) = \mathbf{v} \times (1, 0, 0).$$

Here, $\times$ denotes the cross product.

A20: If A is a matrix with 5 rows and 4 columns such that the set of solutions to the homogeneous system $A\mathbf{x} = 0$ has 2 parameters, what is the rank of A?

A21: Consider the two perpendicular lines through the origin given below:

$$\begin{aligned} L_1 : &\quad x + 2y = 0 \\ L_2 : &\quad x - y/2 = 0 \end{aligned}$$

Find the matrix for the composition of linear transformations: projection onto L_1 followed by projection onto L_2.

A22: Let

$$A = \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}.$$

Compute ABA^T.

A23: Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

A24: A 3×3 matrix `A` with real entries has been typed into MATLAB. The result of the command `[V D] = eig(A)` is (after some slight formatting changes to make it fit better in the exam):

```
V = 0.8165 + 0.0000i   0.8165 + 0.0000i   0.5774 + 0.0000i
    0.0000 + 0.0000i   0.0000 + 0.0000i   0.5774 + 0.0000i
    0.4082 - 0.4082i   0.4082 + 0.4082i   0.5774 + 0.0000i

D = 1.0000 + 2.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
    0.0000 + 0.0000i   1.0000 - 2.0000i   0.0000 + 0.0000i
    0.0000 + 0.0000i   0.0000 + 0.0000i  -1.0000 + 0.0000i
```

Circle *all* true statements below:

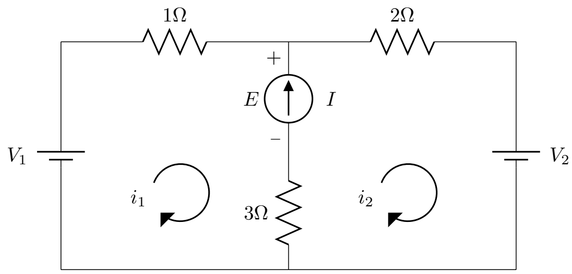
(a) `A` has no real eigenvalues.

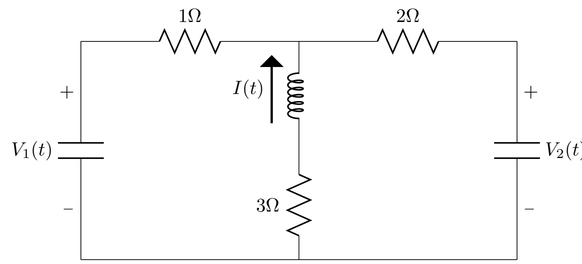
(b) All eigenvalues of `A` have negative real parts.

(c) The eigenvectors of `A` are a basis for $\mathbb{R}^3$.

(d) Eigenvectors of `A` associated to distinct complex eigenvalues are linearly independent.

(e) $[1, 1, 1]^T$ is an eigenvector of `A`.





Questions A25 and A26 concern the circuits above. In the left diagram, I is the current through the current source and E is the voltage across it, to be determined.

A25: For the left circuit above with two voltage sources and one current source write the *one* linear equation that matches the loop currents to the current source.

A26: The left circuit above has solution

$$\begin{aligned} i_1 &= -2I/3 + V_1/3 - V_2/3 \\ i_2 &= I/3 + V_1/3 - V_2/3 \\ E &= 11I/3 + 2V_1/3 + V_2/3 \end{aligned}$$

with all currents in Amps and potentials in Volts. Use this information to derive a differential equation system for $I(t)$, $V_1(t)$, and $V_2(t)$ in the right hand circuit where the two capacitors are 2 Farads and the inductor is 0.1 Henry.

A27: The set of solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$ can be written in parametric form

$$\mathbf{x} = [0, 1, 0, 1]t + [7, 0, 1, 0]s.$$

If A is a 3×4 matrix, what is the reduced row echelon form of A?

A28: Compute $\det(A)$, where A is the 3×3 matrix with complex entries given below. Your answer should be in the form $a + ib$.

$$A = \begin{bmatrix} 2+i & 3-i & 0 \\ 3+i & 2+i & 0 \\ 1 & 1 & i \end{bmatrix}$$

A29: The matrix below represents rotation in 3D about a line through the origin.

$$\begin{bmatrix} 1/2 & -1/\sqrt{2} & 1/2 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/2 & 1/\sqrt{2} & 1/2 \end{bmatrix}$$

Find a vector in the direction of the line of rotation.

A30: A solution to the two component differential equation system $\mathbf{y}' = A\mathbf{y}$ is

$$\mathbf{y}(t) = \begin{bmatrix} i \\ 1 \end{bmatrix} e^{2t}(\cos t + i \sin t).$$

The 2×2 matrix A has real entries. What is A?

Part B - Long Answer Questions, 5 marks each

B1: Consider the lines

$$\begin{aligned} L_1 : &\quad [0, 2, 1] + s[-1, 2, 2] \\ L_2 : &\quad [-1, 0, 3] + t[-2, 1, 1] \end{aligned}$$

(a) [1 mark] Write two distinct points on L_1.

(b) [1] Write a vector that points in the direction parallel to L_1.

(c) [2] Do the lines L_1 and L_2 intersect? If so, find the intersection point. If not, explain.

(d) [1] Find a vector perpendicular to both L_1 and L_2.

B2: In the system below, x, y, and z are variables, and a and b are constants.

$$\begin{array}{ccccccc} x & + & y & + & z & = & 5 \\ x & & & + & z & = & 1 \\ ax & & & + & z & = & b \end{array}$$

(a) [1 mark] Write the system as an augmented matrix.

(b) [2] Bring the augmented matrix to row echelon form.

(c) [1] For what value or values of a and b (if any) does the system have no solutions?

(d) [1] For what value or values of a and b (if any) does the system have an infinite number of solutions?

B3: Consider the 3×3 matrix

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & -4 & 5 \end{bmatrix}.$$

(a) [1 mark] Find an eigenvector of A corresponding to the eigenvalue $\lambda = 1$.

(b) [2] Find all other eigenvalues of A.

(c) [2] Find a basis of eigenvectors of A.

B4: Suppose in the year 2020, 50 million people live in cities and 50 million in the suburbs. Every year, 10% of city residents move to the suburbs and 20% of the residents of the suburbs move to cities.

(a) [1 mark] Write down the 2×2 probability transition matrix P for this problem, using the ordering (1) city and (2) suburbs.

(b) [1] What fraction of residents will be living in cities in 2022?

(c) [2] Find the eigenvalues of P and a basis of eigenvectors.

(d) [1] Assuming the overall population does not change (i.e., remains at 100 million), how many people will be living in the suburbs far in the future?

B5: Consider $z = -1/2 + i\sqrt{3}/2$. Recall that $\tan^{-1}\sqrt{3} = \pi/3$.

(a) [1] Mark the approximate location of z in a sketch of the complex plane.

(b) [1] Compute $|z|$.

(c) [1] Write z in polar form. That is, find a real number $r > 0$ and $0 \leq \theta < 2\pi$ such that $z = re^{i\theta}$.

(d) [2] Find real numbers a and b such that $(-1/2 + i\sqrt{3}/2)^{23} = a + bi$. Simplify your answer.

B6: Consider the three-component differential equation $\mathbf{x}' = A\mathbf{x}$. The 3×3 matrix A has real entries. It has an eigenvalue $\lambda_1 = -2$ and an eigenvalue $\lambda_2 = -1 + i$ with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1+i \\ 1 \end{bmatrix}.$$

(a) [2 marks] Write the general solution to the differential equation.

(b) [2] Write the solution of the differential equation with initial data $\mathbf{x}(0) = [1, 2, 3]^T$.Your solution must be in real form, that is is cannot involve complex numbers or complex exponentials.

(c) [1] Describe all initial conditions for which the solution $\mathbf{x}(t)$ exhibits oscillatory behaviour. Justify your answer briefly.