The University of British Columbia

Final Examination - April 20, 2015

Mathematics 152

All Sections

Closed book examination. No calculators. Time: 2.5 hours

Last Name ____________________ **First** ____________ **Signature** ________________

Section : ________________

Student Number ____________________

Instructor : ________________

Special Instructions:

No books, notes, or calculators are allowed. Show all your work, little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page.

Rules governing examinations

part A		30
B1		5
B2		5
B3		5
B4		5
B5		5
B6		5
Total		60

Part A - Short Answer Questions, 1 mark each

Let $z = 1+3i$ and $u = 1-i$ be the complex numbers in questions A1-A3 below.Your answers should be in the form $a + ib$ where a and b are real numbers.

A1: Compute $z + 2u$.

A2: Compute zu.

A3: Compute z/u.

Points $A = [2, 1]$, $B = [5, 7]$, $C = [0, 4]$ and D form the vertices of a parallelogram $ABCD$ that is the subject of questions A4 and A5 below. *Note:* that B and D are points adjacent to A.

A4: What is the point D?

A5: What is the area of the parallelogram $ABCD$?

A6: Consider the following linear system written in augmented matrix form:

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 5 \\ 0 & 2 & 1 & 7 \end{array}\right]$$

Write the solution or solutions, or state that none exist.

A7: Let $\mathbf{a} = [1, 2, 3]$, $\mathbf{b} = [1, 1, 1]$ and $\mathbf{c} = [1, -4, 1]$. What is the volume of the parallelepiped generated by these vectors?

A8: Let $\mathbf{a} = [1, 2, 3]$, $\mathbf{b} = [1, 1, 1]$ and $\mathbf{c} = [1, -4, 1]$. Find the projection of $\mathbf{a}$ onto $\mathbf{b} + \mathbf{c}$.

A9: Find an eigenvector corresponding to eigenvalue 2 for the matrix below:

$$\begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix}$$

A10: Consider the following lines of MATLAB code:

```
A = zeros(10,10);
for i=1:9
   A(i, i) = 1/2;
   A(i+1, i) = 1/2;
end
A(10,10) =1;
```

Circle the answer below that *best* describes the resulting matrix $\mathbf{A}$:

(a) an error occurs in the last line above.

(b) $\mathbf{A}$ is the transition matrix for a random walk.

(c) $\mathbf{A}$ is an upper triangular matrix.

(d) $\mathbf{A}$ is a diagonal matrix with diagonal entries 1/2.

(e) $\mathbf{A}$ contains the solution of a differential equation system.

A11: Find the parametric form of the plane in $\mathbb{R}^3$ with equation form

$$2x + 3y + 4z = 7$$

A12: Circle *all* possible solution sets for linear systems of five equations in three unknowns.

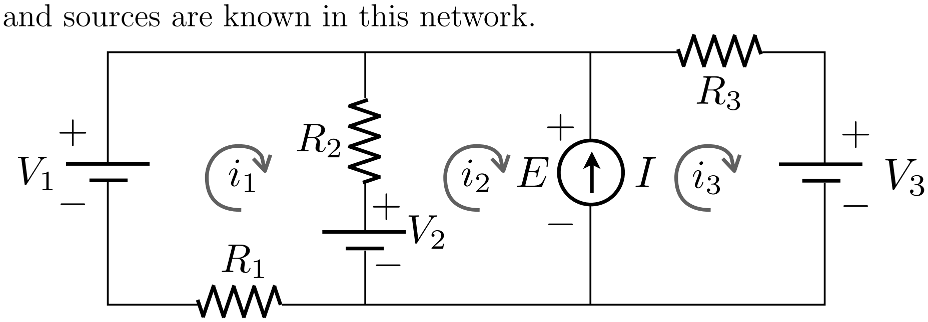
(a) a unique solution.

(b) no solutions.

(c) an infinite number of solutions.

(d) exactly two solutions.

For questions **A13-A15** below consider the resistor network in the diagram below. The resistances and sources are known in this network.



A13: List the unknowns in the linear system for this circuit using the technique of loop currents you learned in the lectures and computer labs this term.

A14: In terms of these unknowns, write the linear equation that represents Kirchhoff's law of voltage drops around the third loop (corresponding to i_3) in the circuit above.

A15: Write a linear equation that expresses the current through the current source in terms of the loop currents in the diagram.

A16: $\mathbf{A}$ is a 2×2 matrix such that

$$\mathbf{A}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{A}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Determine the entries of $\mathbf{A}$.

A17: Consider

$$\mathbf{A} = \begin{bmatrix} 1 & y \\ 2 & z \end{bmatrix}.$$

Find y and z so that $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$ with eigenvalue 2.

A18: Write a matrix whose eigenvalues are the roots of the polynomial

$$z^4 + 3z^3 - 2z^2 + 4 = 0.$$

Do not try to find the roots. *Hint:* Remember your computer lab #6.

A19: Given that the real part x and imaginary part y of the complex number $z = x + iy$ satisfy the equation $(2 - i)x - (1 + 3i)y = 7$, find x and y.

A20: Find the determinant of

$$\begin{bmatrix} 1 & 3 & 2 & 0 & 1 \\ 2 & 7 & 1 & 0 & 2 \\ 1 & 5 & 6 & -1 & 7 \\ 0 & 0 & 3 & 0 & 0 \\ 1 & 1 & 4 & 0 & 4 \end{bmatrix}$$

A21: Let

$$L : \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 3 \\ a \end{bmatrix}$$

be a line in $\mathbb{R}^3$ that is parallel to $2x + 2y + z = 8$. Find a.

A22: Let $z = 2 + 2i$. Write z^5 in the form $a + ib$ with a and b real numbers with no unevaluated trigonometric function values. *Hint:* first write z in polar form.

A23: The matrix

$$\mathbf{A} = \begin{bmatrix} 7 & 0 & -10 \\ 5 & -3 & -5 \\ 5 & 0 & -8 \end{bmatrix}.$$

has the eigenvalue -3. Find all eigenvectors corresponding to this eigenvalue.

A24: Write $\mathbf{b} = [-5, 11, 18]$ as a linear combination $\mathbf{a}_1 = [1, 2, 3]$ and $\mathbf{a}_2 = [3, -1, -2]$ or show it cannot be done.

A25: Find the value for b, for which $\boldsymbol{AB} = \boldsymbol{BA}$, where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} b & 1 \\ 0 & 1 \end{bmatrix}.$$

A26: Find an equation form of the line $[x, y, z] = [3 + 2s, s, 1 - 2s]$.

A27: Find the solution of the system of differential equations

$$\begin{aligned} x' &= 2x + y \\ y' &= x + 2y \end{aligned}$$

that satisfies the initial conditions $x(0) = 2$ and $y(0) = 3$.

A28: Let $\mathbf{A}$ be a 2×2 matrix which represents a reflection across a line in $\mathbb{R}^2$. Suppose that $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is an eigenvector with eigenvalue 1 and $\begin{bmatrix} \alpha \\ 6 \end{bmatrix}$ is an eigenvector with eigenvalue $\beta \neq 1$. What are the values of α and β?

A29: Solve the matrix equation $3\mathbf{A} + \frac{1}{2}\mathbf{B}\mathbf{X} = \mathbf{C}$ for $\mathbf{X}$, in which $\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \mathbf{X}$ are all 2×2 matrices and that

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} -1 & 3 \\ 2 & -11 \end{bmatrix}$$

A30: Consider the probability transition matrix

$$\mathbf{P} = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/3 \\ 1/2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2/3 \end{bmatrix} \quad \text{and initial probability} \quad \mathbf{x}_0 = \begin{bmatrix} 1/11 \\ 3/11 \\ 3/11 \\ 4/11 \end{bmatrix}$$

What is $\lim_{n\to\infty} \mathbf{P}^n \mathbf{x}_0$? *Hint:* this can be done without extensive calculations.

Part B - Long Answer Questions, 5 marks each

B1: Uno, Duo and Traea are three friends. They all owe money to a loan shark. All together they owe $600. Duo owes $200 more than Uno. Uno and Duo combined owe as much as Traea.

(a) [2 marks] Let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

be the vector of unknowns, where x_1, x_2 and x_3 is the amount of money Uno, Duo and Traea owe, respectively. Describe the information above as a linear system in the form

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

(write $\mathbf{A}$ and $\mathbf{b}$ with specific values).

(b) [1] Write the system you found above in augmented matrix form.

(c) [2] Solve the system above using Gaussian elimination on the augmented matrix. How much does each person owe?

B2: Let **A** be the matrix

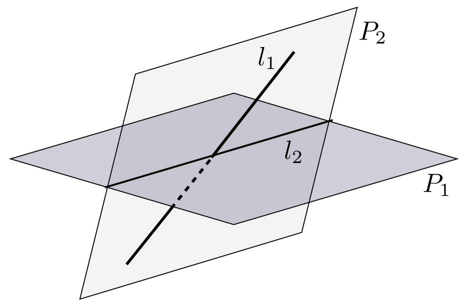
$$\begin{bmatrix} 4 & -1 & 7 \\ 0 & 3 & 0 \\ 1 & 2 & -2 \end{bmatrix}$$

(a) [2 marks] Find an eigenvector of **A** corresponding to eigenvalue $\lambda_1 = 3$.

(b) [2] Find all the other eigenvalues of **A**.

(c) [1] How many linearly independent eigenvectors does **A** have? Justify briefly.

B3: Line l_1 and plane P_1 shown in the figure are given in parametric form as



$$l_1: \ \boldsymbol{x} = t\boldsymbol{a}, \qquad P_1: \ \boldsymbol{x} = \boldsymbol{q} + s_1\boldsymbol{b}_1 + s_2\boldsymbol{b}_2,$$

where

$$\boldsymbol{a} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \qquad \boldsymbol{q} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{b}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{b}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

(a) [2 marks] Find the intersection point of the line l_1 and plane P_1.

(b) [2] Write the parametric form for plane P_2, which contains the line l_1 and is perpendicular to the plane P_1.

(c) [1] Planes P_1 and P_2 intersect along the line l_2. Write the equation form for the line l_2.

B4: Consider the differential equation system

$$\mathbf{x}' = \mathbf{A}\mathbf{x}.$$

where $\mathbf{A}$ has the eigenvalues $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$ with the corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1+i \\ 1-i \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1-i \\ 1+i \end{bmatrix}.$$

(a) [2 marks] Write the general solution of the DE system (in either real or complex form).

(b) [2] Find the solution that satisfies $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in real form (no complex numbers or complex exponentials).

(c) [1] Write MATLAB commands that would compute the first column of the matrix $\mathbf{A}$ from the information given about the eigenvalues and eigenvectors of $\mathbf{A}$.

B5: The percentage of people with the disease, March Madness, is recorded every week. Note that it is possible to recover from March Madness one week and catch it again the following week. Records indicate that the disease can be modelled by a random walk and that if 50% of the population is infected with March Madness one week, then 60% of the population will be infected the next week. Records also indicate that if 100% of the population is infected one week, then 90% of the population will be infected the next week. It is known that 10% of the population has March Madness this week.

(a) [2 marks] What is the 2×2 probability transition matrix for this system?

(b) [1] What percentage of the population will have March Madness two weeks from now?

(c) [1] What percentage of the population had March Madness last week?

(d) [1] Approximately what will be the percentage of people with March Madness many weeks from now?

B6: The matrix

$$\boldsymbol{A} = \begin{bmatrix} 1/2 & -\sqrt{2}/2 & 1/2 \\ \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 1/2 & \sqrt{2}/2 & 1/2 \end{bmatrix}$$

represents a rotation in 3D relative to some axis.

(a) [2 marks] Find all the eigenvalues of $\boldsymbol{A}$.

(b) [1] Find the direction vector of the axis of the rotation. *Hint:* this vector remains unchanged after the rotation.

(c) [2] Find the angle of rotation around the axis in (b). *Hint:* rotate a vector that is perpendicular to the rotation axis.