The University of British Columbia

Math 120 — Honours Calculus

2009, December 18

Instructor: Omer Angel

Name: ______________________________ Student id. ____________________

Instructions

This exam consists of 7 questions worth as noted for a total of 100.
Duration: $2\frac{1}{2}$ hours.
Show all work and calculations and explain your reasoning thoroughly.
Read all questions; They are **not** in order of difficulty.
No calculators or other aids are permitted.
Make sure this exam has 11 pages including this cover page.

Good luck, and happy holydays.

Question	Points	Score
1	42	
2	10	
3	10	
4	12	
5	8	
6	10	
7	8	
Total:	100	

(42 points) 1. **Short-Answer Questions**. Put your answer in the box provided but show your work also. Each question is worth 3 marks. Not all questions are of equal difficulty. Full marks will be given for correct answers placed in the box; at most 1 mark will be given for incorrect answers. Unless otherwise stated, it is not necessary to simplify your answers in this question.

(a) Evaluate $f'(x)$ if $f = e^{x^2+\cos x}$.

Answer

(b) Evaluate $\lim_{x\to 1} \frac{x^3 - e^{x-1}}{\sin(\pi x)}$

Answer

(c) Find $f(2)$ if $f'(x) = \pi f(x)$ for all x, and $f(0) = 2$.

Answer

(d) Evaluate $f'(x)$ if $f(x) = \sqrt{\frac{x-1}{x+1}}$.

Answer

(e) Find $\frac{dy}{dx}$ if $xy + e^x + e^y = 1$.

Answer

(f) Use a linear approximation to estimate $\tan^{-1}(1.1)$, using $\tan^{-1} 1 = \pi/4$.

Answer

(g) Let $f(x) = x + \cos x$, and $g(y) = f^{-1}(y)$ be the inverse function. Determine $g'(y)$.

Answer

(h) Find constants a, b so that the following function is differentiable:

$$f(x) = \begin{cases} ax^2 + b & x \leq 1 \\ e^x & x > 1. \end{cases}$$

Answer

(i) Find $\lim_{n\to\infty} \frac{(n+1)^4 \sin n}{n^{6+\sin n}}$.

Answer

(j) Perform one iteration of Newton's method for finding a root of $x - \cos x$, starting with $x_0 = 0$.

Answer

(k) For which values of a is the function $f(x) = \begin{cases} 0 & x \leq 0 \\ x^a \sin(\frac{1}{x}) & x > 0 \end{cases}$ differentiable at 0?

Answer

(l) Find c so that $\lim_{x \to 0} \frac{1+cx-\cos x}{e^{x^2}-1}$ exists.

Answer

(m) Find the point promised by the mean value theorem for the function e^x on the interval $[0, T]$

Answer

(n) Find the limit of $a_n = \sqrt{n^2 + 5n} - n$.

Answer

Long answer questions. For questions 2–7, give detailed and jutified answers.

(10 points) 2. Find the maximal possible volume of a cylinder with surface area A. (The surface area consists of two discs and a $(2\pi r \times h)$ strip.)

(10 points) 3. (a) Use the definition of the limit to prove $\lim_{x \to 2} \frac{1}{x} = \frac{1}{2}$.

(b) Use the formal definition of the derivative to compute $f'(x)$ if $f(x) = \sqrt{1+x}$. (You do not need to use the formal definition of limit.)

(12 points) 4. Consider the function $f(x) = xe^{-x^2/2}$.

(a) Find the limit of f as $x \to \infty$ and $x \to -\infty$.

(b) Find inflection points, intervals of increase, decrease, convexity and concavity. You may use without proof the formula $f''(x) = (x^3 - 3x)e^{-x^2/2}$.

(c) Find local and global minima and maxima.

(d) Use all the above to draw a graph for f. Indicate all special points on the graph.

(8 points) 5. Suppose $f(0) = 0$ and $f'(x) = \dfrac{1}{1 + e^{-f(x)}}$. Prove that $f(100) < 100$.

(10 points) 6. (a) Find P_8: the Taylor polynomial of order 8 for $f(x) = e^{x^2}$ around $x = 0$.

(b) Use this to find $f^{(8)}(0)$.

(c) Give an upper bound on the error $|f(x) - P_8(x)|$ that is valid for all $x \in [-1, 1]$.

(8 points) 7. (a) If in a sequence $(a_0, a_1, \dots)$, each term is the average of the previous two, show that the sequence converges.

(4 bonus) (b) Find the limit in terms of a_0 and a_1.