

The University of British Columbia

Math 120 — Honours Calculus

2009, December 18

Instructor: Omer Angel

Name: _____ Student id. _____

Instructions

This exam consists of 7 questions worth as noted for a total of 100.

Duration: $2\frac{1}{2}$ hours.

Show all work and calculations and explain your reasoning thoroughly.

Read all questions; They are **not** in order of difficulty.

No calculators or other aids are permitted.

Make sure this exam has 11 pages including this cover page.

Good luck, and happy holydays.

1. Each candidate should be prepared to produce his library/AMS card upon request.
2. Read and observe the following rules: No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions. CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action. (a) Making use of any books, papers or memoranda, other than those authorized by the examiners. (b) Speaking or communicating with other candidates. (c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.
3. Smoking is not permitted during examinations.

Question	Points	Score
1	42	
2	10	
3	10	
4	12	
5	8	
6	10	
7	8	
Total:	100	

(42 points) 1. **Short-Answer Questions.** Put your answer in the box provided but show your work also. Each question is worth 3 marks. Not all questions are of equal difficulty. Full marks will be given for correct answers placed in the box; at most 1 mark will be given for incorrect answers. Unless otherwise stated, it is not necessary to simplify your answers in this question.

(a) Evaluate $f'(x)$ if $f = e^{x^2 + \cos x}$.

Answer

(b) Evaluate $\lim_{x \rightarrow 1} \frac{x^3 - e^{x-1}}{\sin(\pi x)}$

Answer

(c) Find $f(2)$ if $f'(x) = \pi f(x)$ for all x , and $f(0) = 2$.

Answer

(d) Evaluate $f'(x)$ if $f(x) = \sqrt{\frac{x-1}{x+1}}$.

Answer

(e) Find $\frac{dy}{dx}$ if $xy + e^x + e^y = 1$.

Answer

(f) Use a linear approximation to estimate $\tan^{-1}(1.1)$, using $\tan^{-1} 1 = \pi/4$.

Answer

(g) Let $f(x) = x + \cos x$, and $g(y) = f^{-1}(y)$ be the inverse function. Determine $g'(y)$.

Answer

(h) Find constants a, b so that the following function is differentiable:

$$f(x) = \begin{cases} ax^2 + b & x \leq 1 \\ e^x & x > 1. \end{cases}$$

Answer

(i) Find $\lim_{n \rightarrow \infty} \frac{(n+1)^4 \sin n}{n^6 + \sin n}$.

Answer

(j) Perform one iteration of Newton's method for finding a root of $x - \cos x$, starting with $x_0 = 0$.

Answer

- (k) For which values of a is the function $f(x) = \begin{cases} 0 & x \leq 0 \\ x^a \sin(\frac{1}{x}) & x > 0 \end{cases}$ differentiable at 0?

Answer

- (l) Find c so that $\lim_{x \rightarrow 0} \frac{1+cx-\cos x}{e^{x^2}-1}$ exists.

Answer

- (m) Find the point promised by the mean value theorem for the function e^x on the interval $[0, T]$

Answer

- (n) Find the limit of $a_n = \sqrt{n^2 + 5n} - n$.

Answer

Long answer questions. For questions 2–7, give detailed and justified answers.

- (10 points) 2. Find the maximal possible volume of a cylinder with surface area A . (The surface area consists of two discs and a $(2\pi r \times h)$ strip.)

- (10 points) 3. (a) Use the definition of the limit to prove $\lim_{x \rightarrow 2} \frac{1}{x} = \frac{1}{2}$.
- (b) Use the formal definition of the derivative to compute $f'(x)$ if $f(x) = \sqrt{1+x}$. (You do not need to use the formal definition of limit.)

(12 points) 4. Consider the function $f(x) = xe^{-x^2/2}$.

- (a) Find the limit of f as $x \rightarrow \infty$ and $x \rightarrow -\infty$.
- (b) Find inflection points, intervals of increase, decrease, convexity and concavity. You may use without proof the formula $f''(x) = (x^3 - 3x)e^{-x^2/2}$.
- (c) Find local and global minima and maxima.
- (d) Use all the above to draw a graph for f . Indicate all special points on the graph.

(8 points) 5. Suppose $f(0) = 0$ and $f'(x) = \frac{1}{1 + e^{-f(x)}}$. Prove that $f(100) < 100$.

- (10 points) 6. (a) Find P_8 : the Taylor polynomial of order 8 for $f(x) = e^{x^2}$ around $x = 0$.
(b) Use this to find $f^{(8)}(0)$.
(c) Give an upper bound on the error $|f(x) - P_8(x)|$ that is valid for all $x \in [-1, 1]$.

(8 points) 7. (a) If in a sequence (a_0, a_1, \dots) , each term is the average of the previous two, show that the sequence converges.

(4 bonus) (b) Find the limit in terms of a_0 and a_1 .