Name: ______________ Student Number: ______________

Math 120 Final Exam December 2007 2.5 hours.

There are **12 pages** in this test including this cover page. **No calculators, books, notes, or electronic devices of any kind are permitted. Unless otherwise indicated, show all your work.**

Rules governing formal examinations:

1. Each candidate must be prepared to produce his/her library/AMS card upon request;
2. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions;
3. No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination;
4. Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action;
   (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
   (b) Speaking or communicating with other candidates;
   (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received;
5. Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator; and
6. Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

Problem #	Value	Grade
1	42	
2	16	
3	12	
4	6	
5	6	
6	8	
7	10	
Total	100	

I have read and understood the instructions and agree to abide by them.

Signed: ______________

1. ([42 marks]) **Short-Answer Questions.** Put your answer in the box provided but show your work also. Each question is worth 3 marks. Full marks will be given for correct answers placed in the box, but at most 1 mark will be given for incorrect answers. Unless otherwise stated, it is not necessary to simplify your answers in this question.

   (a) Evaluate $\lim_{x\to 2} \frac{x^2+x-6}{x-2}$.

   (b) Evaluate $f'(2)$ if $f(x) = \ln(g(xh(x)))$, $h(2) = 2$, $h'(2) = 3$, $g(4) = 3$, $g'(4) = 5$.

   (c) Evaluate $\lim_{x\to -\infty} \frac{1-x-x^2}{2x^2-7}$.

(d) Find the values of the constants a and b for which

$$f(x) = \begin{cases} \cos(x) & x \leq 0 \\ ax + b & x > 0 \end{cases}$$

is differentiable everywhere.

(e) Find the derivative of $e^{\cos(x^2)}$.

(f) For the curve defined by the equation $\sqrt{xy} = x^2 y - 2$, find the slope of the tangent line at the point $(1, 4)$.

(g) If $f(x) = \sin(x^2)$, compute $f^{(6)}(0)$. *Hint*: it may help to use Maclaurin polynomials.

(h) Find the (x, y) coordinates of all points where the graph of the parametric curve $x = \sin(t^2)$, $y = \cos(t^2)$ has a vertical tangent.

(i) If $f(x) = (\cos x)^{\sin x}$, find $f'(x)$.

(j) Use a linear approximation to estimate $(2.001)^3$. Write your answer in the form $n/1000$ where n is an integer.

(k) $f(x) = 2x - \sin(x)$ is one-to-one. Find $(f^{-1})'(\pi - 1)$.

(l) A point is moving on the unit circle $\{ (x, y) \mid x^2 + y^2 = 1 \}$ in the xy-plane. At $(2/\sqrt{5}, 1/\sqrt{5})$, its y-coordinate is increasing at rate 3. What is the rate of change of its x-coordinate?

(m) Find the function $y(t)$ if $\frac{dy}{dt} + 3y = 0$, $y(1) = 2$.

(n) For the function

$$f(x) = \begin{cases} 0 & x \leq 0 \\ \frac{\cos(x) - 1}{\sqrt{x}} & x > 0 \end{cases},$$

write in the box the (roman) number of the correct statement from the list:

i. f is undefined at $x = 0$
ii. f is neither continuous nor differentiable at $x = 0$
iii. f is continuous but not differentiable at $x = 0$
iv. f is differentiable but not continuous at $x = 0$
v. f is both continuous and differentiable at $x = 0$

Full-Solution Problems. In questions 2-7, justify your answers and **show all your work**.

2. ([16 marks]) Let $f(x) = x\sqrt{3-x}$.

   (a) ([2 marks]) Find the domain of $f(x)$.

Answer

   (b) ([4 marks]) Determine the x-coordinates of the local maxima and minima (if any) and intervals where $f(x)$ is increasing or decreasing.

   (c) ([2 marks]) Determine intervals where $f(x)$ is concave upwards or downwards, and the x coordinates of inflection points (if any). You may use, without verifying it, the formula $f''(x) = (3x-12)(3-x)^{-3/2}/4$.

· · · question continued on next page

(d) ([2 marks]) There is a point at which the tangent line to the curve $y = f(x)$ is vertical. Find this point.

Answer

(e) ([2 marks]) The graph of $y = f(x)$ has no asymptotes. However, there is a real number a for which $\lim_{x \to -\infty} \frac{f(x)}{|x|^a} = -1$. Find the value of a.

Answer

(f) ([4 marks]) Sketch the graph $y = f(x)$, showing the features given in items (a) to (d) above and giving the (x, y) coordinates for all points occuring above.

3. ([12 marks]) What is the largest possible area of a window, with perimeter P, in the shape of a rectangle with a semicircle on top (so the diameter of the semicircle equals the width of the rectangle)?

4. ([6 marks]) Find an equation of a line that is tangent to both of the curves $y = x^2$ and $y = x^2 - 2x + 2$ (at different points).

Answer

5. ([6 marks]) Let $f(x) = x|x|$.

(a) *Using the definition of the derivative*, show that $f(x)$ is differentiable at $x = 0$.

(b) Find the second derivative of $f(x)$. Explicitly state, with justification, the point(s) at which $f''(x)$ does not exist, if any.

6. ([8 marks]) Use the definition of limit to prove that $\lim_{x \to 3} x^2 = 9$.

7. ([10 marks]) Let $f(x) = \sqrt{x}$.

(a) Find the third-order Taylor polynomial for f around $x = 1$.

(b) Evaluate

$$\lim_{t \to 0} \frac{f(1+t) - \cos(t/2) - \sin(t/2)}{t^3}.$$