The University of British Columbia

Final Examination - April 24, 2017

Mathematics 105

All Sections

Closed book examination Time: 2.5 hours

Last Name ____________ **First** ________ **Signature** ______________

Student Number ________ **Section Number** ________ **Instructor** ____________

Special Instructions:

No books, notes, or calculators are allowed. A formula sheet is included.

1a,b,c,d,e		15
1f,g,h,i,j		15
1k,l,m,n		12
2		12
3		6
4		12
5		20
6		8
Total		100

[42] **1. Short-Answer Questions**. Put your answer in the box provided but show your work also. Each question is worth 3 marks, but not all questions are of equal difficulty.

(a) Find an equation of the plane that passes through the point $(0, -3, 2)$ with a normal vector $\langle 2, -1, 4 \rangle$.

Answer:

(b) Let $z = 2\cos(x + y)$. Is the point $(\pi, 3\pi)$ on the level curve $z = 2$?

Answer:

(c) Let $s(x, z) = z^2 \tan xz$. Find $\dfrac{\partial s}{\partial z}$.

Answer:

(d) Let $f(x, y) = -4x^2 + 8y^2 - 3$. Find the critical point(s) of $f(x, y)$, and determine (if possible) whether each critical point corresponds to a local maximum, local minimum, or saddle point.

Answer:

(e) Does a left Riemann sum underestimate or overestimate the area of the region under the graph of a function that is positive and decreasing on an interval $[a, b]$?

Answer:

(f) Let $f(x) = \displaystyle\int_{-x}^{x} \sqrt[3]{1 + t^2}\, dt$. Find the derivative $f'(x)$ of $f(x)$.

Answer:

(g) Evaluate $\displaystyle\int_{-2}^{1} x^3 e^{x^4+1}\,dx$.

Answer:

(h) Evaluate $\displaystyle\int \sin^{-1} y\,dy$, where $\sin^{-1} y = \arcsin y$.

Answer:

(i) Evaluate $\displaystyle\int \sin^5 x\,dx$.

Answer:

(j) Evaluate the limit of the sequence $\left\{\ln\left(\sin\left(\frac{3}{n}\right)\right) + \ln n\right\}_{n=1}^{\infty}$.

Answer:

(k) Evaluate $\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^{k}\left(2^{3-k}\right)$ or state that it diverges.

Answer:

(l) Find the first three nonzero terms of the Maclaurin series for the function $f(x) = \tan^{-1}\left(\frac{x}{3}\right)$.

Answer:

(m) Let $f(x) = \begin{cases} k & \text{for } -2 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$. Find the constant k such that $f(x)$ is a probability density function.

Answer:

(n) Find the cumulative distribution function of the probability density function $g(x) = e^{-x}e^{-e^{-x}}$, where $-\infty < x < \infty$.

Answer:

Full-Solution Problems. In questions 2 – 6, justify your answers and show all your work.

[12] **2**. (a) Evaluate $\displaystyle\int \frac{3x^2 + 2x + 8}{4x^2 - x^3}\,dx$.

2.(b) Evaluate $\displaystyle\int \frac{(x-4)^2}{(9+8x-x^2)^{3/2}}\,dx.$

[6] **3**. Use the method of Lagrange Multipliers to find the maximum value of the utility function $U = f(x, y) = 16x^{\frac{1}{4}}y^{\frac{3}{4}}$, subject to the constraint $G(x, y) = 50x+100y-500,000 = 0$, where $x \geq 0$ and $y \geq 0$. You do not need to justify your answer. Note that a solution that does not use the method of Lagrange Multipliers will receive no credit, even if the answer is correct.

[12] **4.**

(a) If we use Simpson's Rule to approximate the integral $\int_1^2 x(1-\ln x)\,dx$, how large should n be so that the error is not larger than $\frac{1}{900000}$?

(b) Solve the following initial value problem:

$$y' = te^{-t}y^2, \quad y(0) = 1.$$

[20] **5.**

(a) Suppose $\{a_n\}$ is a sequence, and its associated sequence of partial sum $s_N = \sum_{n=1}^{N} a_n$ is given by $s_N = 3 - \frac{N}{2N+1}$. Evaluate $\lim_{n\to\infty} a_n + \sum_{n=1}^{\infty} a_n$.

(b) Let $\sum_{n=0}^{\infty} c_n x^n$ be the Maclaurin series for $f(x) = \frac{4}{1+2x} + \frac{1}{1+x}$, i.e., $\sum_{n=0}^{\infty} c_n x^n = \frac{4}{1+2x} + \frac{1}{1+x}$. Find c_n for all n.

5.(c) Determine whether the series $\sum_{n=10}^{\infty} \frac{e^n}{n!}$ converges or diverges.

5.(d) Determine whether the series $\sum_{k=1}^{\infty} \frac{k(2+\sin k)}{k^{\sqrt{2}}}$ converges or diverges.

5.(e) Find the radius of convergence of the power series $\displaystyle\sum_{k=1}^{\infty} \frac{(x-1)^{2k}}{k(5^k)}$.

[8] **6.**

(a) Determine if there exist two continuous functions $f : [0,1] \longrightarrow \mathbb{R}$ and $g : [0,1] \longrightarrow \mathbb{R}$ such that

$$\int_0^1 f(x)g(x)\,dx = 0, \quad \text{but} \quad \int_0^1 f(x)\,dx \neq 0 \quad \text{and} \quad \int_0^1 g(x)\,dx \neq 0.$$

If your answer is YES, please give the two functions f and g explicitly by some formulas. If your answer is NO, explain why.

(b) Evaluate

$$\lim_{x\to 0} \frac{1}{x} \int_0^{2017x} \sqrt{2017 + t^{2017}}\,dt.$$

MATH 105 Exam Formula Sheet

- **Summation formulas:**

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

- **Trigonometric formulas:**

$$\cos^2 x = \frac{1+\cos(2x)}{2}, \quad \sin^2 x = \frac{1-\cos(2x)}{2}, \quad \sin(2x) = 2\sin x\cos x$$

- **Derivatives of some inverse trigonometric functions:**

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

- **Simpson's rule:**

$$S_n = \frac{\Delta x}{3}\Big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n)\Big)$$

$$\frac{K(b-a)(\Delta x)^4}{180} \geq E_S, \qquad K \geq |f^{(4)}(x)| \quad \text{on } [a,\, b]$$

- **Indefinite integrals:**

$$\int \sec x dx = \ln|\sec x + \tan x| + C$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + C = \arctan x + C$$

- **Probability:**

If X is a continuous random variable with probability density function $f(x)$ with $-\infty < x < \infty$, then the expected value $\mathbf{E}(X)$, the variance $Var(X)$ and the standard deviation $\sigma(X)$ are given by

$$\mathbf{E}(X) = \int_{-\infty}^{\infty} x f(x)dx$$

$$Var(X) = \int_{-\infty}^{\infty} \big(x - \mathbf{E}(X)\big)^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - \big(\mathbf{E}(X)\big)^2$$

$$\sigma(X) = \sqrt{Var(X)}$$

- **Some commonly used Taylor series centered at 0:**

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \qquad \text{for } |x| < 1$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \qquad \text{for } |x| < \infty$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \qquad \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \qquad \text{for } |x| < \infty$$

$$\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \qquad \text{for } 1 \geq |x|$$

- **Two important limits:**

$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \qquad \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$

The End