

The University of British Columbia

Final Examination - April 24, 2017

Mathematics 105

All Sections

Closed book examination

Time: 2.5 hours

Last Name _____ First _____ Signature _____

Student Number _____ Section Number _____ Instructor _____

Special Instructions:

No books, notes, or calculators are allowed. A formula sheet is included.

Senate Policy: Conduct during examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (a) speaking or communicating with other candidates, unless otherwise authorized;
 - (b) purposely exposing written papers to the view of other candidates or imaging devices;
 - (c) purposely viewing the written papers of other candidates;
 - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1a,b,c,d,e		15
1f,g,h,i,j		15
1k,l,m,n		12
2		12
3		6
4		12
5		20
6		8
Total		100

[42] 1. **Short-Answer Questions.** Put your answer in the box provided but show your work also. Each question is worth 3 marks, but not all questions are of equal difficulty.

- (a) Find an equation of the plane that passes through the point $(0, -3, 2)$ with a normal vector $\langle 2, -1, 4 \rangle$.

Answer:

- (b) Let $z = 2 \cos(x + y)$. Is the point $(\pi, 3\pi)$ on the level curve $z = 2$?

Answer:

- (c) Let $s(x, z) = z^2 \tan xz$. Find $\frac{\partial s}{\partial z}$.

Answer:

- (d) Let $f(x, y) = -4x^2 + 8y^2 - 3$. Find the critical point(s) of $f(x, y)$, and determine (if possible) whether each critical point corresponds to a local maximum, local minimum, or saddle point.

Answer:

- (e) Does a left Riemann sum underestimate or overestimate the area of the region under the graph of a function that is positive and decreasing on an interval $[a, b]$?

Answer:

- (f) Let $f(x) = \int_{-x}^x \sqrt[3]{1+t^2} dt$. Find the derivative $f'(x)$ of $f(x)$.

Answer:

(g) Evaluate $\int_{-2}^1 x^3 e^{x^4+1} dx$.

Answer:

(h) Evaluate $\int \sin^{-1} y dy$, where $\sin^{-1} y = \arcsin y$.

Answer:

(i) Evaluate $\int \sin^5 x dx$.

Answer:

(j) Evaluate the limit of the sequence $\left\{ \ln \left(\sin \left(\frac{3}{n} \right) \right) + \ln n \right\}_{n=1}^{\infty}$.

Answer:

(k) Evaluate $\sum_{k=0}^{\infty} \left(\frac{1}{4} \right)^k (2^{3-k})$ or state that it diverges.

Answer:

- (l) Find the first three nonzero terms of the Maclaurin series for the function $f(x) = \tan^{-1}\left(\frac{x}{3}\right)$.

Answer:

- (m) Let $f(x) = \begin{cases} k & \text{for } -2 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$. Find the constant k such that $f(x)$ is a probability density function.

Answer:

- (n) Find the cumulative distribution function of the probability density function $g(x) = e^{-x}e^{-e^{-x}}$, where $-\infty < x < \infty$.

Answer:

Full-Solution Problems. In questions 2 – 6, justify your answers and show all your work.

[12] **2.** (a) Evaluate $\int \frac{3x^2 + 2x + 8}{4x^2 - x^3} dx$.

2.(b) Evaluate $\int \frac{(x-4)^2}{(9+8x-x^2)^{3/2}} dx$.

[6] **3.** Use the method of Lagrange Multipliers to find the maximum value of the utility function $U = f(x, y) = 16x^{\frac{1}{4}}y^{\frac{3}{4}}$, subject to the constraint $G(x, y) = 50x + 100y - 500,000 = 0$, where $x \geq 0$ and $y \geq 0$. You do not need to justify your answer. Note that a solution that does not use the method of Lagrange Multipliers will receive no credit, even if the answer is correct.

[12] 4.

(a) If we use Simpson's Rule to approximate the integral $\int_1^2 x(1 - \ln x) dx$, how large should n be so that the error is not larger than $\frac{1}{900000}$?

(b) Solve the following initial value problem:

$$y' = te^{-t}y^2, \quad y(0) = 1.$$

[20] 5.

(a) Suppose $\{a_n\}$ is a sequence, and its associated sequence of partial sum $s_N = \sum_{n=1}^N a_n$ is

given by $s_N = 3 - \frac{N}{2N+1}$. Evaluate $\lim_{n \rightarrow \infty} a_n + \sum_{n=1}^{\infty} a_n$.

(b) Let $\sum_{n=0}^{\infty} c_n x^n$ be the Maclaurin series for $f(x) = \frac{4}{1+2x} + \frac{1}{1+x}$, i.e., $\sum_{n=0}^{\infty} c_n x^n = \frac{4}{1+2x} + \frac{1}{1+x}$. Find c_n for all n .

5.(c) Determine whether the series $\sum_{n=10}^{\infty} \frac{e^n}{n!}$ converges or diverges.

5.(d) Determine whether the series $\sum_{k=1}^{\infty} \frac{k(2 + \sin k)}{k^{\sqrt{2}}}$ converges or diverges.

5.(e) Find the radius of convergence of the power series $\sum_{k=1}^{\infty} \frac{(x-1)^{2k}}{k(5^k)}$.

[8] **6.**

- (a) Determine if there exist two continuous functions $f : [0, 1] \rightarrow \mathbb{R}$ and $g : [0, 1] \rightarrow \mathbb{R}$ such that

$$\int_0^1 f(x)g(x) dx = 0, \quad \text{but} \quad \int_0^1 f(x) dx \neq 0 \quad \text{and} \quad \int_0^1 g(x) dx \neq 0.$$

If your answer is YES, please give the two functions f and g explicitly by some formulas.
If your answer is NO, explain why.

- (b) Evaluate

$$\lim_{x \rightarrow 0} \frac{1}{x} \int_0^{2017x} \sqrt{2017 + t^{2017}} dt.$$

MATH 105 Exam Formula Sheet

- **Summation formulas:**

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4}$$

- **Trigonometric formulas:**

$$\cos^2 x = \frac{1 + \cos(2x)}{2}, \quad \sin^2 x = \frac{1 - \cos(2x)}{2}, \quad \sin(2x) = 2 \sin x \cos x$$

- **Derivatives of some inverse trigonometric functions:**

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

- **Simpson's rule:**

$$S_n = \frac{\Delta x}{3} \left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n) \right)$$

$$\frac{K(b-a)(\Delta x)^4}{180} \geq E_S, \quad K \geq |f^{(4)}(x)| \quad \text{on } [a, b]$$

- **Indefinite integrals:**

$$\int \sec x dx = \ln |\sec x + \tan x| + C$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + C = \arctan x + C$$

- **Probability:**

If X is a continuous random variable with probability density function $f(x)$ with $-\infty < x < \infty$, then the expected value $\mathbf{E}(X)$, the variance $Var(X)$ and the standard deviation $\sigma(X)$ are given by

$$\mathbf{E}(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$Var(X) = \int_{-\infty}^{\infty} (x - \mathbf{E}(X))^2 f(x) dx = \int_{-\infty}^{\infty} x^2 f(x) dx - (\mathbf{E}(X))^2$$

$$\sigma(X) = \sqrt{Var(X)}$$

- **Some commonly used Taylor series centered at 0:**

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad \text{for } |x| < 1$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \text{for } |x| < \infty$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

$$\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \quad \text{for } |x| < 1$$

- **Two important limits:**

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1, \quad \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

The End