The University of British Columbia

Final Examination - April 13, 2011

Mathematics 105, 2010W T2

All Sections

Closed book examination Time: 2.5 hours

Last Name ________________ **First** ____________ **SID** ________________

Section number ______ **Instructor name** ________________

Special Instructions:

1. A separate formula sheet will be provided. No books, notes, or calculators are allowed.
2. Show all your work. A correct answer without accompanying work will get no credit.
3. If you need more space than the space provided, use the back of the previous page. Where boxes are provided for answers, put your final answers in them.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

Q	Points	Max
1		15
2		15
3		15
4		15
5		15
6		10
7		15
8		15
9		35
Total		150

1. (a) Determine the following indefinite integral:

$$\int \cos(\ln x)\, dx.$$

(10 points)

(b) Find the value of the following definite integral:

$$\int_0^{\frac{\pi}{2}} \cos^3 x \sin^2 x \, dx.$$

(5 points)

2. Sketch the region in the first quadrant bounded by the curves

$$y = x, \qquad y = 4x \quad \text{and} \quad xy = 1$$

and find the area of this region.

(5 + 10 = 15 points)

3. Is there any value of k for which the function f below is a probability density function?

$$f(x) = \begin{cases} \frac{2k}{(k+x)(k-x)} & \text{for } 0 \leq x \leq \frac{k}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

If yes, find all such values of k. If there is no such k, explain why.

(15 points)

4. The unit price of a certain commodity can fluctuate anywhere between \$2 and \$6.50. If the unit price is $\$p$, the supplier produces $(p-2)$ units of it, while the consumers demand $65p^{-1} - 10$ units. Find the market equilibrium and the consumer surplus.

(5 + 10 = 15 points)

5. You open a savings account under the Escalator Plan with an initial deposit of $\$P$. The advantage of this plan is that the interest rate is not fixed, but grows proportionally with time as long as the account is alive. In other words, the money in the account collects interest at the annual rate of rt at time t, compounded continuously (here r is a constant). You also keep adding money to the account in the form of a continuous deposit of $\$S(t)$ at time t.

   (a) Write down the initial value problem for the amount $A(t)$ in your account in time t.

   (5 points)

   (b) Now suppose that $P = 100$, $r = 0.01$ and $S(t) = 0$. Under these assumptions, solve the initial value problem you wrote down in part (a).

   (5 points)

(c) With the same conditions as in part (b), how long does it take for the account size to reach $\$100e^2$ after the initial deposit?

(5 points)

6. A certain T-shirt company makes two types of shirts. Type A says "Much Music rocks" while Type B says "Business Calculus rocks". The company produces 1000 T-shirts in total every year. It sells each Type A shirt at \$15 and each type B at \$10. If the cost of producing x shirts of type A and y shirts of type B is

$$C = \frac{1}{200}x^2 + 6x + 4y + 4000,$$

determine how many shirts of each kind the company should produce to maximize its profit every year. Give a simple justification why the answer you obtained gives the maximum and not the minimum.

(9+1 = 10 points)

7. Find numbers a and b such that the function

$$f(x,y) = \ln(xy^2) + bx^2 + 3axy$$

has a critical point at $(1,3)$. Is this critical point a local maximum, a local minimum or a saddle?

(10 + 5 = 15 points)

8. The productivity (measured in the dollar value of goods produced) of a certain country is given by the function

$$f(x, y) = 160x^{\frac{3}{4}}y^{\frac{1}{4}},$$

where x denotes the amount (in dollars) invested in labor, and y is the amount invested in capital. Recall that the marginal productivity of labor (respectively capital) is the rate of change of f with respect to x (respectively y), holding y (respectively x) fixed.

(a) Suppose that the country is currently utilizing 81 units of labor and 16 units of capital. Find the current marginal productivity of labor and also the current marginal productivity of capital.

(5 points)

(b) Suppose that the government has the following two policy options,

- **Policy I:** increase labor by 1 unit without changing capital, and
- **Policy II:** increase labor by $\frac{1}{2}$ units and capital by $\frac{1}{3}$ units.

Using the marginal productivities obtained in part (a), find approximations for the changes in productivity under each of these policies. Based on these approximations, which policy should the government encourage for higher productivity?

(5 points)

(c) Find all vectors $\mathbf{v}$ with the property $|\mathbf{v}| = |\nabla f(81, 16)|$ that point in the direction of no change in productivity from its current value.

(5 points)

9. Each of the short-answer questions below is worth 5 points. Put your answer in the box provided and show your work. No credit will be given for the answer without the correct accompanying work.

   (a) Does

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

   exist? If it does, find its value. If not, write "does not exist" in the box and give reasons why.

Answer:

(b) What is the slope of the curve

$$y = \int_0^{2\sin x} e^{-t^2}\, dt$$

at the point $x = 0$?

Answer:

(c) Express the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln\left(1 + \frac{k}{n}\right)$$

as a definite integral, and leave it in the form of an integral. Do not evaluate it!

Answer:

(d) Determine whether the improper integral

$$\int_0^2 \frac{2x}{1-x^2}\,dx$$

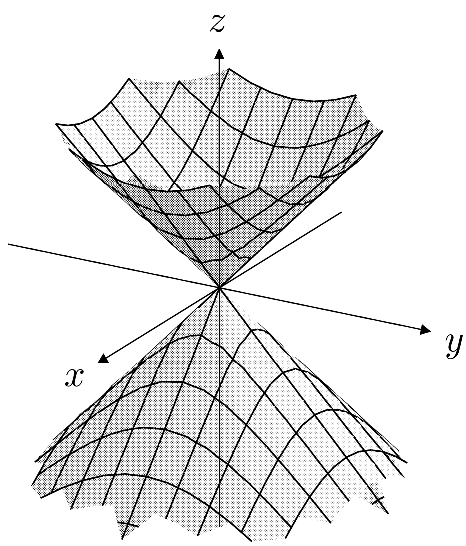
is convergent or divergent. If convergent, write the value of the integral in the box. If not, write "divergent" and explain why in the space provided below.

Answer:

(e) In the box, write down which of the equations

$$z = x^2 + y^2, \quad z = x^2 - y^2, \quad x^2 + y^2 + z^2 = 1, \quad z^2 = x^2 + y^2$$

describes the surface with the following diagram?



Answer:

If $z = f(x, y)$ is the equation of the surface in the diagram, sketch in the space provided below the level curve of f at height $z = 2$. Provide explicit labels for your sketch.

(f) A discrete random variable takes only two values, 0 and 1. Find $p = \Pr(X = 1)$ if the variance of X is $1/4$.

Answer:

(g) You are given a function f satisfying $f(1,0) = 0$, $f_x(1,0) = 3$ and $f_y(1,0) = -1$. Find $z'(0)$ if

$$z(t) = f(e^{3t}, \sin 2t).$$

Answer:

The End