This examination has 18 pages of questions excluding this cover

The University of British Columbia

Final Exam - April 25, 2016

Mathematics 103: Integral Calculus with Applications to Life Sciences

201, 203 (Perkins), 202, 208 (First), 206 (Scurll), 207 (Lopez), 209 (K.F.Li), 212 (D. Li)

Closed book examination Time: 2.5 hours (150 minutes)

Last Name: ____________________ **First Name:** ____________________

Student Number: ____________________ **Section:** ____________________

Rules governing examinations:

1. No books, notes, electronic devices or any papers are allowed. Electronic devices must be turned off.
2. You may continuing writing on the back of the previous page, and there is an extra page at the end of the exam. Additional paper is available upon request.
3. You must be prepared to produce your library/AMS card upon request.
4. No student shall be permitted to enter the examination room after 15 minutes or to leave less than 15 minutes before the completion of the examination. Students must ask invigilators for permission to use the washrooms.
5. You are not allowed to communicate with other students during the examination. Students may not purposely view other's written work nor purposefully expose his/her own work to the view of others or any imaging device.
6. At the end of the exam, you will put away all writing implements upon instruction. Students will continue to follow all of the above rules while the papers are being collected.
7. Students must follow all instructions provided by the invigilators.
8. Students are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions
9. Any deviation from these rules will be treated as an academic misconduct. The plea of accident or forgetfulness shall not be received.

I agree to follow the rules outlined above ______________________________

(signature)

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	10	12	6	8	8	20	12	12	12	100
Score:										

Important

1. Simplify all your answers as much as possible and express answers in terms of fractions or constants such as $\sqrt{e}$ or $\ln(2)$ rather than decimals.
2. Unless otherwise indicated, show all your work and explain your reasonings clearly!
3. Questions in a section are weighted evenly unless otherwise stated.
4. Formula sheet at the back (you may tear it off and use it for scratch work).

1. Short-answer-problems. (Full credit will be given for a correct final answer. For partial credit you must show your work.)

   (a) (2 points) Express $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} e^{-k^2/n^2}$ as a definite integral of the form $\int_0^1 f(x)\mathrm{d}x$. Do not evaluate the integral.

   ANSWER: ______________

   (b) (2 points) Find the limit of the sequence $\left\{ \dfrac{2n^3 + 3\sin(n)}{5n^3 + n} \right\}_{n\geq 1}$.

   ANSWER: ______________

   (c) (2 points) Evaluate $\lim\limits_{x\to\pi} \dfrac{3\sin(x)}{(x-\pi)}$. (Simplify your answer as much as possible.)

   ANSWER: ______________

(d) (2 points) Compute $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\displaystyle\int_0^{3x} e^{\cos t}\mathrm{d}t\right)$ at $x = 0$.

ANSWER: ______________

(e) (2 points) Find the value of C so that $p(x) = Cx^{-3}$ is a probability density function on $[1, \infty)$.

ANSWER: ______________

2. Compute the following integrals:

   (a) (3 points) $\displaystyle\int \frac{1}{x^2+2x+2}\,\mathrm{d}x.$

   ANSWER:______________________________________

   (b) (3 points) $\displaystyle\int x\cos(x)\,\mathrm{d}x.$

   ANSWER:______________________________________

(c) (3 points) $\displaystyle\int_{\pi/12}^{\pi/6} \frac{\cos(3x)}{\sin(3x)}\,\mathrm{d}x.$

ANSWER:____________________________________

(d) (3 points) $\displaystyle\int \arcsin(x)\,\mathrm{d}x.$

ANSWER:____________________________________

3. (6 points) Find the length of the curve $y = 2x^{3/2}$ from $x = 0$ to $x = \frac{1}{3}$.

ANSWER:__________

4. The density of a metal bar 3 meters long is is given by $\rho(x) = \frac{1}{\sqrt{x+1}}$ kilograms per meter $(0 \leq x \leq 3)$.

   (a) (4 points) Find the mass of the bar.

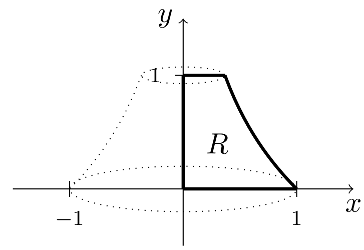
   ANSWER:__________

   (b) (4 points) Find the center of mass of the bar.

   ANSWER:__________

5. Let R be the region bounded between the curve $y = -\ln x$, and the lines $y = 0$, $y = 1$, $x = 0$.

   (a) (4 points) Find the volume of the solid obtained by revolving R around the <u>y-axis</u> (see illustration).



ANSWER:__________

   (b) (4 points) The solid of part (i) is filled with liquid of varying density. It is given that liquid density y units from the bottom of the solid is e^{-y} grams per cubic unit. Find the mass of the liquid inside the solid.

ANSWER:__________

6. (a) (13 points) Consider a population that is governed by the logistic differential equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{1}{2}N\left(1 - \frac{N}{10^6}\right)$$

with initial condition $N(0) = 2 \times 10^5$. N is the population size and t is time measured in years.

(i) [1 point] Write down the carrying capacity, K.

ANSWER:__________

(ii) [1 point] The substitution $y = \frac{N}{K}$, where K is the carrying capacity found in part (ii), rescales the differential equation to

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{2}y(1-y).$$

Write down the initial condition $y(0)$.

ANSWER:__________________

Note: It is possible to do the remaining parts of this question even if you were unable to answer (i) or (ii).

(iii) [2 points] Find all steady states of the differential equation in (ii).

ANSWER:___________________

(iv) [7 points] Solve the differential equation in part (ii) for $y(t)$ with the initial condition $y(0)$ found in part (ii). For full credit you must show your work.

ANSWER:____

(v) [2 points] How long will it take for the population to reach 50% of the carrying capacity?

ANSWER:____________________________

(b) (7 points) An animal population in appropriate units after n years, x_n, satisfies the recursion relation

$$x_{n+1} = \frac{1}{2}x_n(1 - x_n) + x_n$$

and starts with an initial population

$$x_0 = \frac{1}{5}.$$

That is, the sequence $\{x_n\}_{n\geq 0}$ is defined by iterating the map $g(x) = \frac{1}{2}x(1-x) + x$.

(i) [2 points] Find the population after one year.

ANSWER:__________________

(ii) [2 points] Find all fixed points of this map g.

ANSWER:___________________

(iii) [3 points] Classify the above fixed points as stable or unstable. Justify your answers.

ANSWER:____

7. (a) (6 points) For each of the following series, decide whether or not it converges. Justify your answers.

(i) $\sum_{n=1}^{\infty} \frac{\cos^4(n)}{n^2}$.

(ii) $\sum_{n=1}^{\infty} \frac{n^4+3}{2n^4+n}$.

(b) (6 points) Consider the power series: $\sum_{n=1}^{\infty} \frac{2^{-n}}{\sqrt{n}} x^n$.

(i) Find the radius of convergence of the power series.

ANSWER:_______________

(ii) Find all values of x such that the power series converges.

ANSWER:_____________________________

8. (a) (3 points) Find a power series for $\cos(x^2)$ about $x = 0$. State (without proof) for which values of x this power series converges.

ANSWER:______________________________

(b) (3 points) Find $\frac{\mathrm{d}^8}{\mathrm{d}x^8}(\cos(x^2))$ at $x = 0$.

ANSWER:______________________

(c) (3 points) Find the Taylor series for $F(x) = \int_0^x \cos(t^2)\,dt$ about $x = 0$. State (without proof) for which values of x this power series converges.

ANSWER:____

(d) (3 points) Use the first 2 non-zero terms in the above Taylor series to estimate $\int_0^{1/2} \cos(t^2)\,dt$. (Error estimates are not required.)

ANSWER:____________________

9. Let x be a continuous random variable taking values in $[0, 100]$ with probability density function $p(x)$, mean value μ, variance σ^2, median value x_{med}, and cumulative function $F(x)$.

   (a) (2 points) Write down the integral which expresses μ in terms of $p(x)$.

   ANSWER:____

   (b) (2 points) Write down the integral which expresses σ^2 in terms of $p(x)$ and μ.

   ANSWER:____

   (c) (3 points) If $F(1/2) = \frac{1}{8}$, $F(1) = \frac{1}{2}$, $F(2) = \frac{3}{4}$, and $F(3) = \frac{7}{8}$. Find the probability that x is greater than $2x_{\text{med}}$.

   ANSWER:__________

(d) (5 points) Show that $\mu = \int_0^{100} (1 - F(x))\,\mathrm{d}x$.
Hint. The formula for μ in (a) is a good starting point.

Additional space for work

Useful Formulæ

Summation

$$\sum_{k=1}^{N} k = \frac{N(N+1)}{2}$$

$$\sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}$$

$$\sum_{k=1}^{N} k^3 = \left(\frac{N(N+1)}{2}\right)^2$$

Trigonometric identities

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta; \qquad \text{for } \alpha=\beta\text{:} \quad \sin(2\alpha) = 2\sin\alpha\cos\alpha$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta; \qquad \text{for } \alpha=\beta\text{:} \quad \cos(2\alpha) = 2\cos^2\alpha - 1 = \cos^2\alpha - \sin^2\alpha$$

$$\cos^2(\alpha) = \frac{1+\cos(2\alpha)}{2}; \qquad \sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2}$$

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\tan^2\alpha + 1 = \sec^2\alpha = \frac{1}{\cos^2\alpha}$$

Some useful trigonometric values

$$\sin(0) = 0, \quad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}, \quad \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \quad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}, \quad \sin\left(\frac{\pi}{2}\right) = 1, \quad \sin(\pi) = 0$$

$$\cos(0) = 1, \quad \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \quad \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \quad \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, \quad \cos\left(\frac{\pi}{2}\right) = 0, \quad \cos(\pi) = -1$$

Derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan x = \frac{1}{1+x^2}$$