This examination has 13 pages of questions excluding this cover

The University of British Columbia
Final Exam - April 24, 2014

Mathematics 103: Integral Calculus with Applications to Life Sciences

202 (Carlquist), 203 (Kim), 206 (Bruni), 207 (Namazi), 208 (Wong), 209 (Wardil)

Closed book examination Time: 150 minutes

Last Name: ____________________ **First Name:** ____________________

Student Number: ________________ **Section:** circle above

Rules governing examinations:

1. No books, notes, electronic devices or any papers are allowed. To do your scratch work, use the back of the examination booklet. Additional paper is available upon request.
2. You should be prepared to produce your library/AMS card upon request.
3. No student shall be permitted to enter the examination room after 10 minutes or to leave before the completion of the examination.
4. You are not allowed to communicate with other students during the examination. Students may not purposely view other's written work nor purposefully expose his/her own work to the view of others or any imaging device.
5. At the end of the exam, you will put away all writing implements upon instruction. Students will continue to follow all of the above rules while the papers are being collected.
6. Students must follow all instructions provided by the invigilator.
7. Students are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions
8. Any deviation from these rules will be treated as an academic misconduct. The plea of accident or forgetfulness shall not be received.

I agree to follow the rules outlined above ______________________________

(signature)

Question:	1	2	3	4	5	6	7	8	9	10	11	Total
Points:	16	8	10	8	9	10	8	8	8	12	3	100
Score:												

Important

1. Simplify all your answers as much as possible and express answers in terms of fractions or constants such as $\sqrt{e}$ or $\ln(4)$ rather than decimals.
2. Show all your work and explain your reasonings clearly!

1. (16 points) **Multiple Choice Questions – Circle the correct answer.**
   (No need to show work in this question.)

   a. (2 marks) Which of the following is equal to $\frac{d}{dx}\left[\int_0^{x^2} \sin t\, dt\right]$?

   (**A**) $2x\sin(x^2)$ (**B**) $\sin(x^2)$ (**C**) $\cos x$ (**D**) $2x\cos(x^2)$

   b. (2 marks) Let $p(x)$ be a probability density function defined on the interval $[0, 10]$. Determine whether each of the following statements is *True* or *False*:

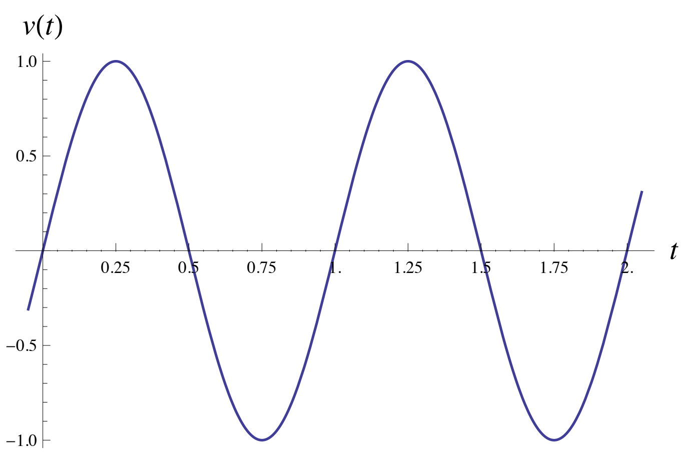
   i. Let $f(x) = \int_2^x p(t)dt$. Then the derivative of $f(x)$ at $x = 5$ satisfies $f'(5) \geq 0$:

   True / False

   ii. $0 \leq \int_0^{10} tp(t)dt \leq 10$:

   True / False

   c. (6 marks) The graph below depicts the velocity of a car traveling along a straight road as a function of time, $v(t) = \sin(2\pi t)$. Initial time is $t = 0$.



   Determine whether each of the following statements is *True, False* or *Indeterminate*:

   i. At $t = 0$ the car starts with a non zero initial velocity:

   True / False / Indeterminate

   ii. The car returns to its initial position when $t = 1$:

   True / False / Indeterminate

   iii. The car is moving away from its original position when $t = 5/4$:

   True / False / Indeterminate

d. (4 marks) Let's assume the size of a microbial population $x(t)$ (in millions) at time t (hours) is determined by the differential equation $\dfrac{dx}{dt} = x(1-x)(x-1/2)$. Circle the correct answers.

i. Determine *all* steady states (or stationary states or equilibria).
Circle all steady states:

(**A**) -2 (**B**) -1 (**C**) $-\dfrac{1}{2}$ (**D**) 0 (**E**) $\dfrac{1}{2}$ (**F**) 1 (**G**) 2

ii. If the initial population is $x(0) = 0$, then the population $x(t)$ will converge to the stationary state $x = 1/2$ as $t \to \infty$:

True / False

iii. If $x(0)$ is in the interval $[0,1]$, then the population will stay inside this interval:

True / False

e. (2 marks) Consider the sequence

$$(a_k)_{k=1}^{\infty} = \left(1, 0, \frac{1}{2}, 0, 0, \frac{1}{3}, 0, 0, 0, \frac{1}{4}, 0, 0, 0, 0, \frac{1}{5}, \dots\right)$$

and its associated series $\sum_{k=1}^{\infty} a_k$. Circle the correct answers.

i. The sequence $\{a_k\}_{k=1}^{\infty}$ converges to 0 as $k \to \infty$:

True / False

ii. The series $\sum_{k=1}^{\infty} a_k$ diverges:

True / False

2. (8 points) (**Work must be shown for full marks. Simplify fully.**)

   a. (3 marks) Compute the arc length of the curve $y = f(x)$ for $1 \leq x \leq 4$ with $f(x) = \int_1^x \sqrt{t^3 - 1}\, dt$

   b. (5 marks) Let
   $$f(x) = x^2, \qquad \text{for } 0 \leq x \leq 1.$$
   Find the volume of the solid generated by revolving $y = f(x)$ (for $0 \leq x \leq 1$) around the horizontal line $y = -1$. *Hint: a sketch might help.*

3. (10 points) (**Work must be shown for full marks. Simplify fully.**)

Vancouver has some of the best salmon hatcheries (i.e. salmon farms) in the world. The probability that a young salmon takes t days to swim from the Capilano Salmon Hatchery to the Pacific Ocean is given by the probability density function

$$p(t) = e^{3-t} \qquad \text{for } 3 \leq t \leq \infty.$$

[It is assumed that it takes at least 3 days for a salmon to reach the ocean.]

a. (3 marks) Find the probability that a young salmon takes <u>at least</u> 5 days to swim from the hatchery to the ocean.

b. (3 marks) Find the average (i.e. mean) time it takes for a young salmon to swim from the hatchery to the Pacific Ocean.

c. (4 marks) Assume that the lifetime of a salmon depends on how quickly it manages to reach the Pacific Ocean as a juvenile. Suppose that a salmon lives s years, where $s = \ln(t+1)$, if it reaches the Pacific Ocean in t days. Find the probability density function $q(s)$ for s.

4. (8 points) (**Work must be shown for full marks. Simplify fully.**)

   a. (3 marks) Evaluate $\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \frac{1}{n}$ if it exists, or show that it does not exist, otherwise.

   b. (3 marks) Let $(a_n)_{n\geq 1}$ be a **sequence**, where $a_n = 1 + \frac{n}{(n+1)!}$. Find the limit of the sequence if it converges, or show that it diverges, otherwise.

   c. (2 marks) Evaluate $\lim_{n\to\infty} n \ln\left(1 + \frac{2}{n}\right)$ if it exists, or show that it does not exist, otherwise.

5. (9 points) **(Work must be shown for full marks. Simplify fully.)**

A mutation occurs within a bacterial species that allows it to acquire antibiotic resistance. Through natural selection the abundance of the mutant trait increases in the population. Let y be the fraction (i.e. $0 \leq y \leq 1$) of the population, which carries the antibiotic resistance trait. Suppose y satisfies the following differential equation:

$$\frac{dy}{dt} = (1-y)^3,$$

where t is time (years).

a. (1 mark) Determine *all* steady state solutions (or stationary solutions or equilibria) of the differential equation.

b. (5 marks) At time $t = 0$ it is determined that half of the bacterial population carries the antibiotic resistance trait. Determine the fraction of the bacterial population with antibiotic resistance as a function of time t.

c. (3 marks) In the same situation as part **b**, how long after $t = 0$ does it take until 80% of the bacterial population is resistant to antibiotics?

6. (10 points) (**Work must be shown for full marks. Simplify fully.**)

   Determine with full justification whether the following series converge. You do not have to evaluate the sums.

   a. (3 marks) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{\cos(e^n)+10}{n^2}\right)$

   b. (3 marks) $\displaystyle\sum_{n=1}^{\infty}\cos^2\left(\frac{1}{n}\right)$

   c. (4 marks) $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k\,(\ln k)^{103}}.$

7. (8 points) (**Work must be shown for full marks. Simplify fully.**)

a. (2 marks) Let $f(x) = e^{-x^2} = \sum_{n=0}^{\infty} a_n x^n$. Determine a_{10} and a_{11}.
(Note: the sum refers to a series expansion centred at $x = 0$.)

b. (2 marks) Let $g(x) = \int_0^x e^{-t^2} dt = \sum_{n=0}^{\infty} b_n x^n$. Determine b_{11} and b_{12}.
(Note: the sum refers to a series expansion centred at $x = 0$.)

c. (4 marks) Suppose the function $y(x) = \sum_{n=0}^{\infty} c_n x^n$ given by a power series solves the following differential equation $\frac{dy}{dx} = 2 + x^2 y$ with the initial value $y(0) = 1$. Determine c_3 and c_4.
Hint: Plug the series into the differential equation.

8. (8 points) (**Work must be shown for full marks. Simplify fully.)**

a. (3 marks) Evaluate $\displaystyle\int_0^{1/2} f(x)\,dx$ where $f(x) = \displaystyle\sum_{n=0}^{\infty}(3n+2)x^{3n+1}$.

(Your final answer should be in the form of a number.)

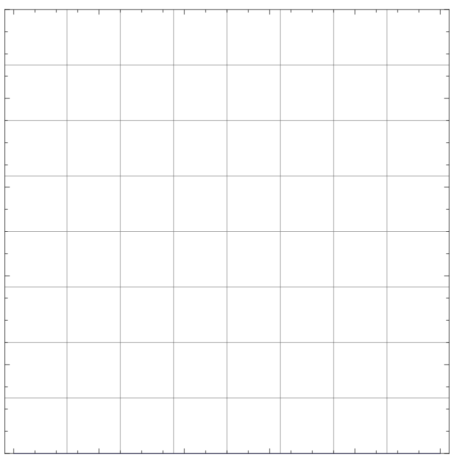
b. (5 marks) Determine **all** values of x such that the following series converges.

$$\sum_{n=0}^{\infty} \frac{(n+1)(x+1)^n}{5^n(n+2)}$$

9. (8 points) Consider the iterated map: $x_{n+1} = F(x_n)$ where $F(x) = \dfrac{3}{2}x(1-x)$ for $0 \leq x \leq 1$.

   a. (4 marks) Find *all* fixed points of the iterated map and determine their stability. *Hint: use $F'(x)$ to check stability.*

   b. (2 marks) Sketch the cobweb corresponding to the sequence x_n with $x_0 = 1/8$.



   c. (2 marks) Suppose a sequence $(a_n)_{n\geq 0}$ satisfies the recursive relation $a_{n+1} = F(a_n)$ with $a_0 = 1/8$. Does the **series** $\displaystyle\sum_{n=0}^{\infty} a_n$ converge? Justify your answer.

10. (12 points) (**Work must be shown for full marks. Simplify fully.**)

a. (4 marks) Evaluate $\displaystyle\int_1^{e^2} \frac{\ln(\sqrt{x})}{\sqrt{x}}\, dx.$

b. (4 marks) Evaluate $\displaystyle\int_0^1 f(x)\, dx$ where $f(x) = \int_1^x \sin(t^2)\, dt.$
Hint: Integration by parts!

c. (4 marks) Consider the following improper integral. Evaluate it if it is convergent, or show that it diverges, otherwise:

$$\int_0^\infty \left(\frac{1}{\sqrt{x^2+1}}\right)^3 dx.$$

11. (3 points) (**This is a difficult question and will be marked very strictly. You should try this ONLY AFTER you finish all other problems.**)

Suppose a sequence $(b_n)_{n\geq 0}$ satisfies the recursive relation

$$b_{n+1} = \beta \frac{b_n}{1+b_n} \ ,$$

where β is a given positive constant. Suppose $b_0 = 1$. Find all **positive** constants β for which the **series** $\sum_{n=0}^{\infty} b_n$ converges. Justify your answer.

Useful Formulas

Summation

$$\sum_{k=1}^{N} k = \frac{N(N+1)}{2} \qquad \sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}$$

$$\sum_{k=1}^{N} k^3 = \left(\frac{N(N+1)}{2}\right)^2 \qquad \sum_{k=0}^{N} r^k = \frac{1-r^{N+1}}{1-r}$$

Trigonometric identities

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta; \qquad \text{for } \alpha=\beta\text{:} \quad \sin(2\alpha) = 2\sin\alpha\cos\alpha$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta; \qquad \text{for } \alpha=\beta\text{:} \quad \cos(2\alpha) = 2\cos^2\alpha - 1$$

$$\sin^2\alpha + \cos^2\alpha = 1 \qquad \tan^2\alpha + 1 = \sec^2\alpha = \frac{1}{\cos^2\alpha}$$

Selected trigonometric values

$$\sin(0) = 0, \quad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}, \quad \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \quad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}, \quad \sin\left(\frac{\pi}{2}\right) = 1, \quad \sin(\pi) = 0$$

$$\cos(0) = 1, \quad \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \quad \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \quad \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, \quad \cos\left(\frac{\pi}{2}\right) = 0, \quad \cos(\pi) = -1$$

Derivatives

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$