This examination has 12 pages of questions excluding this cover

The University of British Columbia
Final Exam - April 19, 2013

Mathematics 103: Integral Calculus with Applications to Life Sciences

201 (Hauert), 203 (Hauert), 206 (Bruni), 207 (Maciejewski), 208 (Ronagh), 209 (Nguyen)

Closed book examination Time: 150 minutes

Last Name: ____________________ **First Name:** ____________________

Student Number: ________________ **Section:** circle above

Rules governing examinations:

1. No books, notes, electronic devices or any papers are allowed. To do your scratch work, use the back of the examination booklet. Additional paper is available upon request.
2. You must be prepared to produce your library/AMS card upon request.
3. No student shall be permitted to enter the examination room after 15 minutes or to leave less than 15 minutes before the completion of the examination. Students must ask the invigilators for permission to use the washrooms.
4. You are not allowed to communicate with other students during the examination. Students may not purposely view other's written work nor purposefully expose his/her own work to the view of others or any imaging device.
5. At the end of the exam, you will put away all writing implements upon instruction. Students will continue to follow all of the above rules while the papers are being collected.
6. Students must follow all instructions provided by the invigilator.
7. Students are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions
8. Any deviation from these rules will be treated as an academic misconduct. The plea of accident or forgetfulness shall not be received.

I agree to follow the rules outlined above ______________________________
(signature)

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	12	12	12	10	10	10	12	10	6	6	100
Score:											

Important

1. Simplify all your answers as much as possible and express answers in terms of fractions or constants such as $\sqrt{e}$ or $\ln(4)$ rather than decimals.
2. Show all your work and explain your reasoning clearly!

1. (12 points) **Sequences & Series:** Short Answer Problems - Determine whether the following sequences and series converge (full marks for correct answer *with justification*; work must be shown for partial marks).

   a. Does the sequence $\left(\dfrac{k^k}{k!}\right)_{k\geq 1}$ converge? If it does, calculate limit.

   Diverges: ☐ Converges: ☐ to ______________________________

   b. Does the sequence $\left(\dfrac{n^3+2}{\sqrt{n^6+3}\,+\,\sqrt[3]{27n^9+1}}\right)_{n\geq 1}$ converge? If it does, calculate limit.

   Diverges: ☐ Converges: ☐ to ______________________________

   c. Does the sequence $(a_k)_{k\geq 1}$ with $a_k = \displaystyle\sum_{n=1}^{k} \frac{\pi}{k}\sin\left(\frac{\pi n}{k}\right)$ converge? If it does, calculate limit.

   Diverges: ☐ Converges: ☐ to ______________________________

d. Does the sequence $\left(\dfrac{1}{n\,(\ln n)}\right)_{n\geq 2}$ converge? If it does, calculate limit.

Diverges: ☐ Converges: ☐ to ______________________

e. Does the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\,(\ln n)}$ converge?

Diverges: ☐ Converges: ☐ ______________________

f. Determine all x such that the series $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^n(n+5)}$ converges.

Series converges for x: ______________________

2. (12 points) **Integration:** Short Answer Problems - Evaluate the following integrals; state if a definite integral does not exist; use limits for improper integrals
(full marks for correct answer; work must be shown for partial marks).

a. $I_a = \displaystyle\int_1^{e^2} \frac{\ln x}{x}\, dx.$

$I_a =$ ____________________

b. $I_b = \displaystyle\int \arctan\left(\frac{1}{x}\right)\, dx.$

$I_b =$ ____________________

c. $I_c = \int_0^2 \frac{1}{x^2 - 1}\, dx.$

$I_c =$ ____________________

d. $I_d = \int \sin(\ln x)\, dx.$

$I_d =$ ____________________

3. (12 points) **Continuous probability:**

a. Let $X \geq 0$ be a random variable on the interval $[0, a]$ with probability density function $p(x)$. The mean of X is given by $\bar{x} = \int_0^a x\, p(x)\, dx$. Show that $\bar{x}$ can also be computed as $\bar{x} = \int_0^a (1 - F(x))\, dx$, where $F(x)$ is the cumulative distribution function (*cdf*) of X.

b. Consider $f(x) = \dfrac{x}{\sqrt{1-x}}$. Convert $f(x)$ into a probability density function (*pdf*), $p(x)$, on an interval (a, b).

i. With $p(x) = c \cdot f(x)$, find the minimal a, maximal b and the constant $c > 0$.

ii. Find the mean of $p(x)$.

4. (10 points) **Torricelli's Trumpet:**

   a. Find the volume of the solid obtained from revolving $y = 1/x$ about the x-axis, where x has domain $[1, \infty)$.

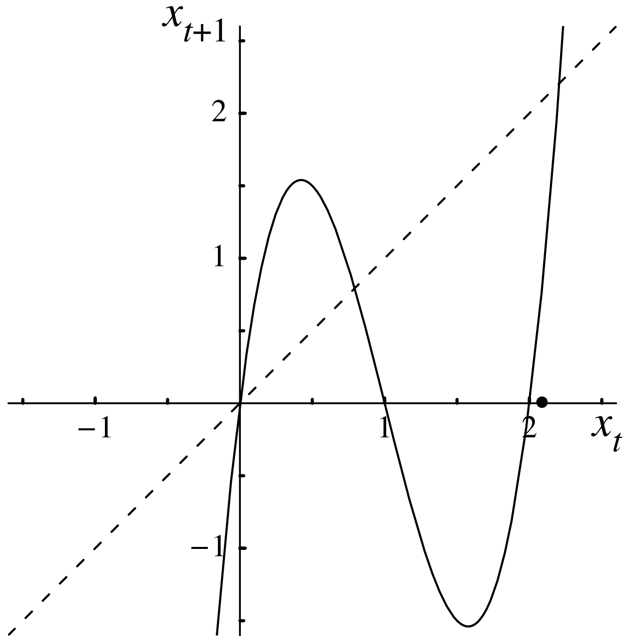
   b. The surface area of a solid of revolution obtained from revolving $y = f(x)$ about the x-axis for $a \leq x \leq b$ is given by

$$A = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

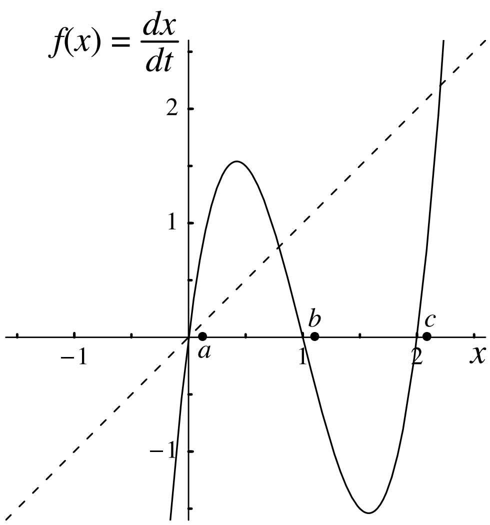
   Consider the area of the surface obtained from revolving $y = 1/x$ about the x-axis, where x has domain $[1, \infty)$. Show whether this surface area exists (integral converges) or diverges. If it exists, provide an upper bound.

5. (10 points) **Dynamical Systems:**

a. Consider the iterated map $x_{t+1} = 4x_t^3 - 12x_t^2 + 8x_t$. Calculate the steady-states (fixed-points, equilibria) and mark them in the sketch below. Determine the stability of each equilibrium (explain analytically or graphically). Plot a cobweb starting with x_0 just above 2 (point indicated in sketch).

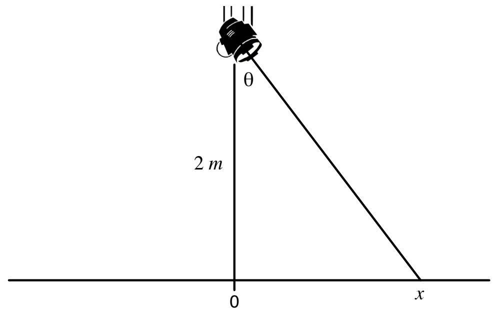


b. Consider the differential equation $\dfrac{dx}{dt} = f(x) = 4x^3 - 12x^2 + 8x$. Calculate the steady-states (equilibria) and mark them in the sketch below. Find $x(t)$ for $t \to \infty$ for the initial points labeled a, b and c, (see sketch).



6. (10 points) **Light show on dance floor:**

A laser is mounted 2 m above the floor (see sketch) and its beam is directed along a straight line on the dance floor. For visual effects the angle, θ, changes periodically and randomly. The angle θ is chosen uniformly in the interval $(-\frac{\pi}{4}, \frac{\pi}{4})$, i.e. the probability density function (*pdf*) is $p(\theta) = \frac{2}{\pi}$ for $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$ and zero elsewhere.



a. Find the cumulative distribution function (*cdf*), $F(x)$, of the x-coordinate of the laser point on the floor.

$F(x) =$ ____________________

b. Find the probability density function (*pdf*), $p(x)$, of the x-coordinate of the laser point on the floor.

$p(x) =$ ____________________

c. The mean x-coordinate, $\bar{x}$, of the laser point is given by $\bar{x} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x\, p(x)\, dx$. Evaluate this integral (or state that it diverges).

$\bar{x} =$ ____________________

7. (12 points) **Biofuel:** Suppose you work at an experimental algae farm where algae is turned into biofuel. The algae grows in tanks at a rate $\alpha = 2$ and the steady supply of nutrients allows each tank to sustain K kilograms of algae.

   a. The rate of change of the total mass of algae, M, is given by:

   $$\frac{dM}{dt} = 2M\left(1 - \frac{M}{K}\right). \tag{*}$$

   Find all steady state solutions (equilibria) for this equation.

   b. Suppose you harvest algae proportional to its total mass at a rate h. Modify equation $(*)$ accordingly. Find all steady state solutions (equilibria) for the new equation. *Hint:* $-h\,M$.

   c. What is the highest rate h^* at which you can harvest algae?

   d. Now suppose that you require a constant supply of algae, so you harvest it by removing algae at a constant rate $H \geq 0$. Modify equation $(*)$ accordingly. Find all steady state solutions (equilibria) for this equation.

   e. In scenario (d), assuming you start with $M = K/4$ kilograms of algae, at what rate H do you have to harvest the algae such that the tank always contains $K/4$ kilograms of algae?

8. (10 points) **Fundamental Theorem of Calculus:**

a. Evaluate and simplify $I_a = \dfrac{d}{dx}\displaystyle\int_x^7 \cos(\sin t)\,dt$.

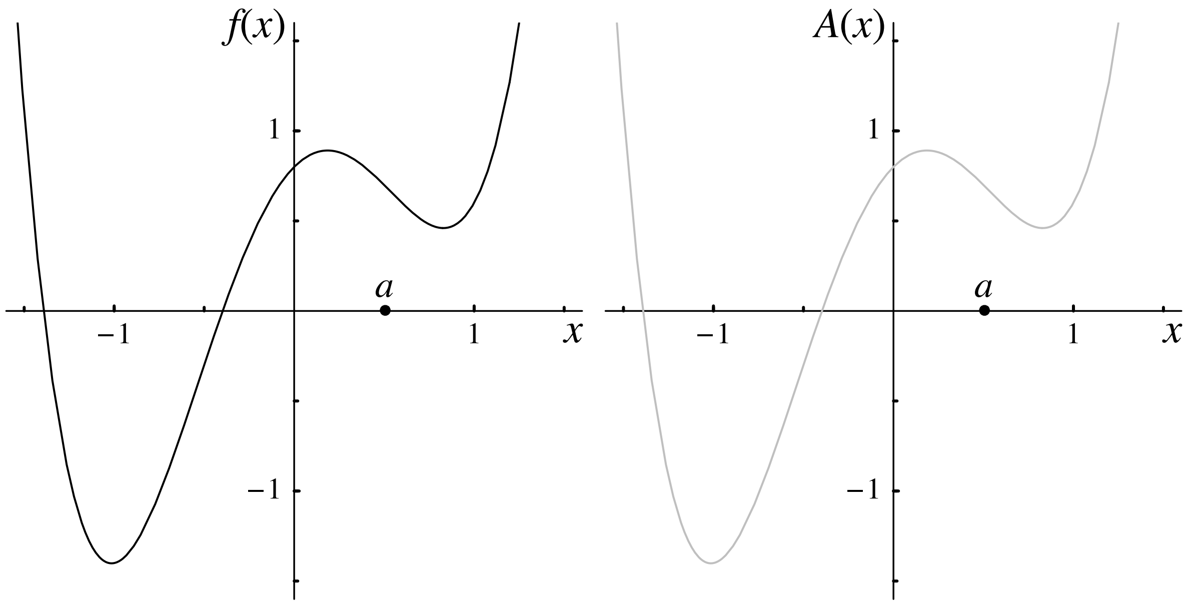
$I_a =$ ____

b. Evaluate and simplify $I_b = \dfrac{d}{dx}\displaystyle\int_1^3 e^{4^t}\,dt$.

$I_b =$ ____

c. Find $\displaystyle\lim_{x\to 0} \frac{1}{x^3}\int_0^x \frac{t^2}{1+t^6}\,dt$.

9. (6 points) **Sketching Antiderivatives:** Consider the sketch of $f(x)$ below (left). Provide a sketch of the antiderivative $A(x) = \int_a^x f(s)\,ds$ for $-1.5 < x < 1.5$ in the panel below (right, $f(x)$ is shown in pale grey as a visual guide). Make sure to clearly indicate local maxima and minima of $A(x)$.



10. (6 points) **Taylor Series:** Find the first five terms of the Taylor series centered at 0 for $F(x) = \int_0^x \ln(1+s^2)\,ds$, i.e. the polynomial $T_4(x)$ with x^4 the highest power of x.

Useful Formulæ

Summation

$$\sum_{k=1}^{N} k = \frac{N(N+1)}{2}$$

$$\sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}$$

$$\sum_{k=1}^{N} k^3 = \left(\frac{N(N+1)}{2}\right)^2$$

$$\sum_{k=0}^{N} r^k = \frac{1-r^{N+1}}{1-r}$$

Trigonometric identities

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta; \qquad \text{for } \alpha=\beta: \quad \sin(2\alpha) = 2\sin\alpha\cos\alpha$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta; \qquad \text{for } \alpha=\beta: \quad \cos(2\alpha) = 2\cos^2\alpha - 1 = \cos^2\alpha - \sin^2\alpha$$

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\tan^2\alpha + 1 = \sec^2\alpha = \frac{1}{\cos^2\alpha}$$

Some useful trigonometric values

$$\sin(0) = 0, \quad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}, \quad \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \quad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}, \quad \sin\left(\frac{\pi}{2}\right) = 1, \quad \sin(\pi) = 0$$

$$\cos(0) = 1, \quad \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \quad \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \quad \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, \quad \cos\left(\frac{\pi}{2}\right) = 0, \quad \cos(\pi) = -1$$

Derivatives

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\tan x = \sec^2 x$$