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Mathematics 102 - Dec 2016
Final Exam — Duration 150 minutes

- Do not circle the boxes. Use **dark pen/pencils** to indicate your choice.

  – Good   – Bad. Will not get identified as your chosen option.

  – Too light.   – Avoid. Might or might not get identified as your chosen option.

- Do not write or mark in the shaded areas labelled *'For marker use only'* nor in the area around the dots in the corners of each page.

Please encode your student number below	Please write your details below
☐0 ☐0 ☐0 ☐0 ☐0 ☐0 ☐0 ☐0	First name:
☐1 ☐1 ☐1 ☐1 ☐1 ☐1 ☐1 ☐1	
☐2 ☐2 ☐2 ☐2 ☐2 ☐2 ☐2 ☐2	Last name:
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☐5 ☐5 ☐5 ☐5 ☐5 ☐5 ☐5 ☐5	Student number:
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☐9 ☐9 ☐9 ☐9 ☐9 ☐9 ☐9 ☐9	

This exam consists of 16 questions worth a total of 60 points.

Instructions

- For multiple choice questions, fill in the box next to your chosen option. Note the box-filling guidelines above. Do not make any marks in a box that you do not want to choose (e.g. to eliminate options as you read through them).
- For all written-answer question, justifying your answers and showing your work is required for getting points. When a box is provided, place your answer in it.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

If you need more space

- There are blank pages at the end. You must indicate to the marker on the page where the question is asked where to look for additional work.

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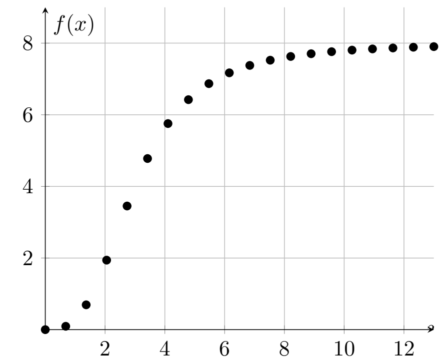
Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
   (i) speaking or communicating with other examination candidates, unless otherwise authorized;
   (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
   (iii) purposely viewing the written papers of other examination candidates;
   (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
   (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Multiple-choice questions

Instructions: For all multiple-choice questions, fill in the box next to the best/correct answer. Each MC question is worth 2 pts.

MC 1 Suppose that you have conducted an experiment resulting in the data plotted below.



The following functions $f(x)$ depend on a parameter $b > 0$. Which function would you use to fit the data points above?

- ☐ $f(x) = \dfrac{8e^x}{b + e^x}$
- ☐ $f(x) = \dfrac{8x^3}{b^3 + x^3}$
- ☐ $f(x) = \dfrac{8x}{b + x}$
- ☐ $f(x) = \dfrac{bx^3}{8^3 + x^3}$

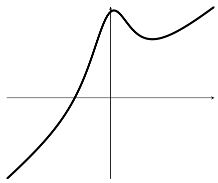
MC 2 A rumour started by one student spreads through UBC according to the model

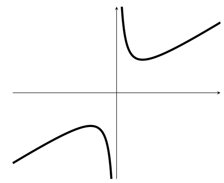
$$I' = 10I - 0.5I^2,$$

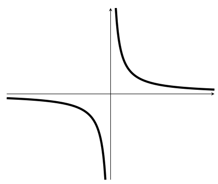
in which $I(t)$ denotes the number of students who have heard the rumour by time t. Which of the following is the best estimate for the number of students who eventually will hear the rumour?

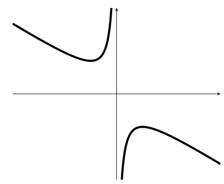
☐ 10 ☐ 200 ☐ 0.5 ☐ 20

MC 3 Which of the following graphs is a sketch of the function $f(x) = x + \frac{1}{x}$ for $x \neq 0$?

☐ 

☐ 

☐ 

☐ 

MC 4 The body temperature $T(t)$ of an individual over a day is a rhythmic process. It reaches a maximum of 37.6° C at 6:00 am and a minimum of 36.8° C at 6:00 pm. Choose the trigonometric function that best describes $T(t)$, where t is given in hours with $t = 0$ taken at midnight (one minute after 11:59 pm).

☐ $T(t) = 37.2 - 0.4\cos\left(\frac{2\pi t}{12}\right)$

☐ $T(t) = 37.2 + 0.4\sin\left(\frac{2\pi t}{24}\right)$

☐ $T(t) = 37.2 + 0.4\cos\left(\frac{2\pi t}{12}\right)$

☐ $T(t) = 37.2 - 0.4\cos\left(\frac{2\pi t}{24}\right)$

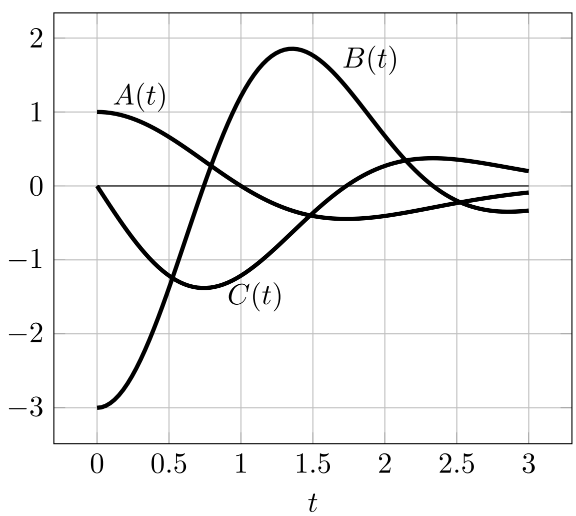
☐ $T(t) = 37.2 + 0.4\sin\left(\frac{2\pi t}{12}\right)$

☐ $T(t) = 37.2 + 0.4\cos\left(\frac{2\pi t}{24}\right)$

MC 5 An object is placed on the kitchen table at time $t = 0$ (t in minutes). Let A be the average rate of change of the temperature of the object over the interval $5 \leq t \leq 12$. Let $T(t)$ denote the temperature of the object at time t. Which of the following statements is true?

- ☐ If the object is a pizza that was taken out of the oven, then $T'(5) > A$.
- ☐ If the object is a bottle of milk that was taken out of the fridge, then $|T'(5)| > |A|$.
- ☐ If the object is a pizza that was taken out of the oven, then $|T'(12)| > |A|$.
- ☐ If the object is a bottle of milk that was taken out of the fridge, then $T'(12) > A$.

MC 6 The graphs below represent the position, velocity and acceleration of a child swinging on a swing. Identify the correct relationships between these functions.



- ☐ $B''(t) = C'(t) = A(t)$
- ☐ $B''(t) = A'(t) = C(t)$
- ☐ $A''(t) = C'(t) = B(t)$
- ☐ $A''(t) = B'(t) = C(t)$
- ☐ $C''(t) = A'(t) = B(t)$
- ☐ $C''(t) = B'(t) = A(t)$

MC 7 Let $f(x)$ be a function that is continuous at $x = 0$. Which of the following statements is correct? Recall that if there exists any function for which a statement is false then the statement is false.

- ☐ If $f'(0)$ exists and $f'(0) = 0$ then $x = 0$ is an extremum.
- ☐ If $x = 0$ is an extremum then $f'(0)$ exists and $f'(0) = 0$.
- ☐ If $f'(x)$ exists for all x and $f'(0) = 0$, $f'(1) > 0$, and $f'(-1) < 0$ then $x = 0$ is a minimum.
- ☐ If $f'(0)$, $f''(0)$ and $f'''(0)$ exist and $f'(0) = f''(0) = 0 \neq f'''(0)$ then $x = 0$ is not an extremum.
- ☐ None of the above are true.

MC 8

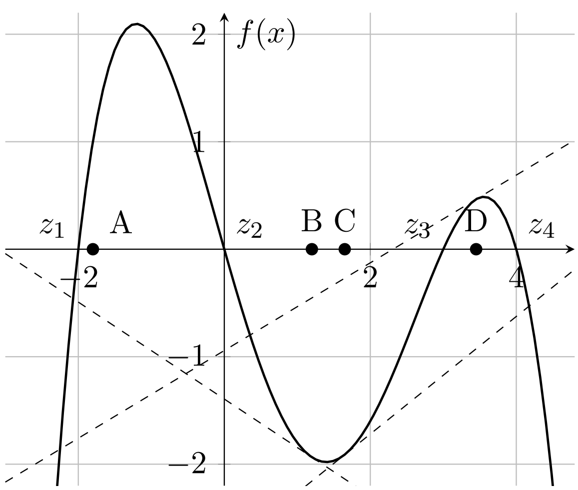
- ☐ Newton's method finds the zero closest to the initial guess x_0.
- ☐ The function $\cos(x)$ has a tangent line at some $x \in [-\pi, \pi]$ that goes through the point $(0, 2)$.
- ☐ A tangent line to the graph of $f(x)$ at an inflection point must cross from one side to the other side of the graph of $f(x)$.
- ☐ If neither $f(x)$ nor $g(x)$ are differentiable at $x = a$ then the function $h(x) = f(x) + g(x)$ cannot be differentiable at a.

Written-answer questions

Instructions: For all written-answer question, justifying your answers and showing your work is required for getting points. When a box is provided, place your answer in it.

Written-answer 1 ☐0 ☐1 ☐2 ☐3 ☐4 — **for marking.**

The zeros of the quartic function $f(x)$ are labeled from z_1 to z_4, running left to right. Using Newton's method, starting at the points $x_0 = A, B, C$, and D (black dots), which zero will be found? Several relevant tangent lines have been provided (dashed lines). Write your answers in the table provided. No justification is required for this problem.



A	
B	
C	
D	

Written-answer 2 ☐0 ☐1 ☐2 ☐3 — **for marking.**

Consider the differential equation

$$\frac{dy}{dt} = 1 - 2y$$

with initial condition $y(0) = 2$. Use Euler's Method with a single step size of $h = \Delta t = \frac{1}{8}$, estimate the value of the solution at $t = \frac{1}{8}$.

$y_1 =$ ☐

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Written-answer 3 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 — **for marking.**

The velocity of a protein inside a cell starting at the top and falling under the force of gravity is given by the differential equation

$$\frac{dv}{dt} = g - kv$$

where $g \approx 10^7 \mu\text{m/s}^2$ is the gravitational acceleration, kv is the acceleration due to drag and $k = 10^{13}\ \text{s}^{-1}$. (Note that this model ignores diffusion of the protein which makes it a really bad model but we're exploring what gravity alone would do to a protein.)

(a) (2 pts) What is the steady state velocity, v_{ss}, of the protein? Include units in your answer.

$v_{ss} =$

(b) (1 pt) For a cell 10 μm in length from top to bottom, how long would it take (in seconds) for gravity alone to pull a protein across the cell if the protein moved at v_{ss} the whole time?

$t_{cross} =$

(c) (4 pts) If the protein is initially at rest at the top of the cell, how long does it take for the protein to reach a velocity of $(1 - e^{-2})v_{ss}$ (roughly 85% of the way to steady state)? You do not have to solve the equation but instead simply state the solution.

$t_{85\%} =$

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Written-answer 4 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 — **for marking.**

The velocity of a particle is determined by its position so that the position function $x(t)$ satisfies the equation $x' = rx(1-x^2)$ where $r > 0$. The particle is at $x(0) = 3$ initially.

(a) (1 pt) What is the speed of the particle at $t = 0$?

$v(0) =$ ☐

(b) (3 pts) Sketch the phase line for the equation and determine what happens to the position of the particle as $t \to \infty$?

$\lim\limits_{t\to\infty} x(t) =$ ☐

(c) (1 pt) Suppose that at $t = 1$s, the particle is at $x(1) = 2$. Another particle whose position $x_2(t)$ is determined by the same equation is initially at $x_2(0) = -3$. What is the location of this particle at $t = 1$s? Hint: Look at the phase line carefully. Explain briefly.

$x_2(1) =$ ☐

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Keep this area clear

Keep this area clear

Written-answer 5 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 — **for marking.**

(a) Show that for $f(x) = \arctan(x)$, we have that $f'(x) = \dfrac{1}{1+x^2}$.

(b) Use linear approximation at a suitable close value to estimate $\arctan(0.9)$. Your answer may be left in terms of fractions.

$\arctan(0.9) \approx$ ☐

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Written-answer 6 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8 — **for marking.**

For both parts of this question, consider the function

$$f(x) = 4x^3 - 12x^2 + 9x.$$

(a) (5 pts) Find the absolute maximum of the function $f(x)$ on the interval $[0, 2]$. Hint: It would be useful to make a rough sketch of $f(x)$.

$f_{max} =$ ☐

Continued on next page...

(b) (3 pts) The absolute maximum of the function $g_p(x) = f(x - p)$ on the interval $[0, 2]$ is a number that depends on p and can be thought of as a function. Call that function $M(p)$ and calculate it for $p \in [0, 2]$. You can use function notation in your answer (e.g. $f(x)$ instead of $4x^3 - 12x^2 + 9x$). Hint: sketch $g_p(x)$ for several values of p.

$$M(p) = \begin{cases} & \text{for} \quad 0 \le p \le \\ & \text{for} \quad\quad < p \le 2 \end{cases}$$

Written-answer 7

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 — **for marking.**

Each day 100,000 people use Vancouver's public transit system. Translink estimates that 2000 of those people ride on transit without paying (called free riders). The number of free riders caught by Translink officers in any particular day is given by $R(N) = 2000(1 - e^{-kN})$ where $k = 1/500$ and N is the number of Translink officers on duty that day. Each officer on duty costs Translink 1000 dollars per day.

(a) (1 pts) The fine collected from each free rider caught is F dollars and Translink hires N officers. Write the function $I(N)$ showing net revenue from fines and cost of hiring?

$I(N) =$ ☐

(b) (3 pts) How many officers should be hired in order to maximize the net revenue for a fixed fine F? Check that your answer is in fact a maximum.

$N =$ ☐

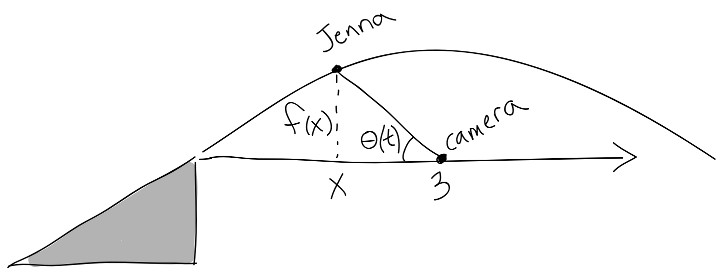
(c) (1 pt) For what value of F, the fine, is it optimal to hire exactly one officer?

$F_1 =$ [ ]

Written-answer 8

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 — **for marking.**

Jenna is riding off a snowboarding ramp while her father takes a video of her flight. The ramp tilts up with a slope of 1/2. Her father stands 3 meters in front of the ramp with his camera pointed at the ramp and held at the same level as the top of the ramp. The angular elevation of the camera, $\theta(t)$, starts at 0 and increases with time so that the camera follows Jenna. When Jenna is a horizontal distance x from the ramp, she is a vertical distance $f(x)$ above the top end of the ramp. She is moving horizontally at a speed of 2 m/s as she leaves the ramp. How rapidly must Jenna's father be rotating the camera so as to follow her flight path just as she leaves the ramp ($x = 0$)? Note that $f(0) = 0$ and $f'(0) = 1/2$.



$\theta'(t) =$

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