Math 102- Final examination

University of British Columbia

December 6, 2014, 3:30 pm to 6:00 pm

Last name (print): First name:

ID number:

Section number:

This exam is "closed book". Calculators or other electronic aids are not allowed.

A		16
B.1		9
B.2		12
B.3		8
C.1		7
C.2		11
C.3		7
Total		70

Rules governing examinations

• Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.

• Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.

• No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.

• Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.

• Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:

(a) speaking or communicating with other candidates, unless otherwise authorized;

(b) purposely exposing written papers to the view of other candidates or imaging devices;

(c) purposely viewing the written papers of other candidates;

(d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,

(e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).

• Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.

• Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.

• Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

A. Multiple choice questions

Enter your choice for each multiple choice question **in the box at the bottom of the page**. There are two pages at the end of the exam that can be used for rough work. No partial marks will be given for this section.

1. Suppose that $g(x)$ is the inverse function of $f(x)$. The tangent line to $f(x)$ at $x = c$ is $y = a + b(x - c)$ where a, b and c are constants. Which of the following is the equation of a tangent line to $g(x)$?

   (A) $y = c + \frac{1}{b}(x - a)$ (B) $y = a + \frac{1}{b}(x - c)$ (C) $y = \frac{1}{a} + \frac{1}{b}\left(x - \frac{1}{c}\right)$

   (D) $y = c + b(x - a)$ (E) $y = c - \frac{1}{b}(x - a)$

2. What value of a makes the function below continuous? Hint: Writing down the definition of the derivative of $\sin(x)$ at $x = 0$ might be useful here.

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x} & \text{if } x \neq 0; \\ a & \text{if } x = 0. \end{cases}$$

   (A) $a = 0$ (B) $a = 1$ (C) $a = 1/2$ (D) $a = \sin(0)/0$ (E) $a = \pi/2$

Enter your answers to these questions here:

MC.1 **[2 pts]**	MC.2 **[2 pts]**

Multiple choice (continued)

3. Consider a differential equation $dy/dt = f(y)$. Shown in A-D is the phase line (state space) diagram ($f(y)$ versus y). Which of the following is the correct pairing of these sketches with the sketch of a solution $y(t)$ to the differential equation?

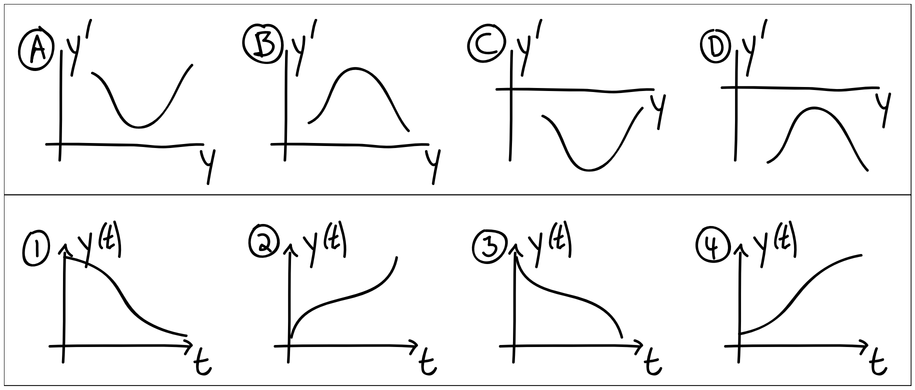
   (a) A1, B4, C2, D3.

   (b) A2, B4, C1, D3.

   (c) A2, B4, C3, D1.

   (d) A4, B2, C3, D1.

   (e) A4, B2, C1, D3.



4. Which differential equation below is the appropriate one for the slope field shown in the figure?

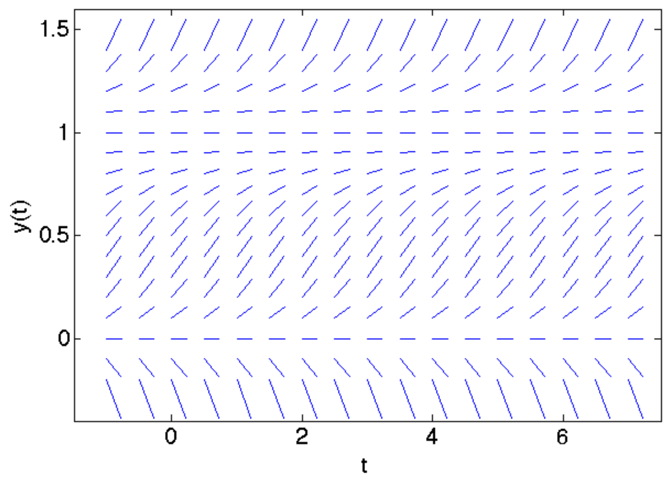
   (a) $y' = y(y-1)$

   (b) $y' = y(1-y)$

   (c) $y' = y(1-y)^2$

   (d) $y' = y^2(1-y)$

   (e) $y' = y^2$



5. The model given below on the left has been suggested for the spread of HIV within the immune system of an infected person. $C(t)$ is the density of healthy immune cells, $I(t)$ is the density of HIV-infected immune cells and $V(t)$ is the density of virus in the blood of a patient. Which of the options on the right gives a correct interpretation of some part of the model?

$$\frac{dC}{dt} = P - \alpha CV - \gamma_1 C$$

$$\frac{dI}{dt} = \alpha CV - \gamma_2 I$$

$$\frac{dV}{dt} = \beta I - \gamma_3 V$$

   (a) Healthy cells can become infected when they encounter infected cells.

   (b) Healthy cells can become infected when they encounter virus.

   (c) Virus is produced at a rate proportional to the current viral density.

   (d) Infected cells die at a rate proportional to the viral density.

   (e) Virus is killed/removed at a rate proportional to the density of healthy immune cells.

Enter your answers to these questions here:

MC.3 [**2 pts**]	MC.4 [**2 pts**]	MC.5 [**2 pts**]

Multiple choice (continued)

6. The figure below shows $\ln(C(t))$ plotted against t as the best-fit line through the data points. Which of the following expressions for $C(t)$ corresponds to that best-fit line?

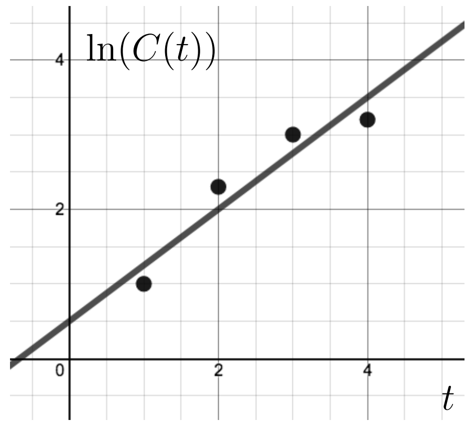
(a) $C(t) = \frac{1}{2} + \frac{3}{4}t$

(b) $C(t) = \frac{3}{4} + \frac{1}{2}t$

(c) $C(t) = \frac{1}{2}e^{\frac{3}{4}t}$

(d) $C(t) = e^{\frac{1}{2}}e^{\frac{3}{4}t}$

(e) $C(t) = e^{\frac{3}{4}}e^{\frac{1}{2}t}$



7. Shown in the figure below are three graphs of temperature over time, T(t), for objects satisfying Newton's Law of Cooling, $dT/dt = k(E - T)$ with $k > 0$. Which of the following is a reasonable description of the three plots?

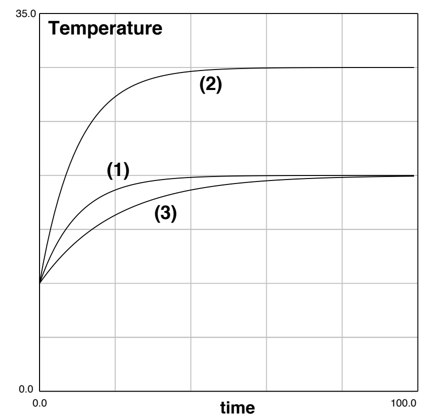
(a) The three objects were all kept in the same ambient temperature E.

(b) The three objects had different initial temperatures.

(c) The three objects had the same constant k.

(d) The constant k for Object 1 was smaller than the value of k for Object 3.

(e) None of the above.



8. In order to calculate arcsin(0.4) using Newton's method, which of the following would be the appropriate formula?

(A) $\arcsin(0.4) \approx 0 + 0.4$

(B) $\arcsin(0.4) \approx \arcsin(0) + \arccos(0)(0.4 - 0)$

(C) $x_{n+1} = x_n - \dfrac{\sin(x_n) - 0.4}{\cos(x_n)}$

(D) $x_{n+1} = x_n - \dfrac{\arcsin(x_n) - 0.4}{\arccos(x_n)}$

(E) None of the above.

Enter your answers to these questions here:

MC.6 **[2 pts]**	MC.7 **[2 pts]**	MC.8 **[2 pts]**

B. Short-Answer Problems

A correct answer in the box will get full points. Partial marks might be given for work shown.

1. **[5 pt]** List the x-coordinates of all the minima, maxima and inflection points of the function $f(x) = e^{-x^2+x}$.

Minima: [ ]

Maxima: [ ]

Inflection points: [ ]

2. **[4 pt]** A population of zebra mussels, an invasive species now found in many rivers in North America, grows at a rate proportional to the current population level starting from $P(0) = 100$. The constant of proportionality is $k = 0.5\ln(2)$ per year. After how many years will $P = 800$?

$P(t) = 800$ when $t =$ [ ]

3. **[4 pt]** Calculate the equation of the tangent line ($L(x)$) to the function $f(x) = \tan(x)$ at the point $x = \pi/4$.

$L(x) =$ [ ]

4. **[5 pt]** The daily temperature in Vancouver during the month of March varies from an average low of 2°C to an average high of 10°C. It is coolest just before sunrise which is roughly 7 AM. Construct a function (**using cosine**) that provides a good description of the temperature $T(t)$ throughout a day in March where t is measured in hours from 12 AM.

$T(t) =$ [ ]

5. **[3 pt]** Consider the differential equation $\frac{dy}{dt} = 2(1 - y)$ with the initial condition $y(0) = 2$. Using Euler's method with a single step of size $h = \Delta t = 1/4$, estimate the value of the solution at $t = 1/4$.

$y(1/4) \approx$ [ ]

6. **[3 pt]** Use linear approximation to estimate $\ln(0.95)$. Recall that $\ln(1) = 0$.

$\ln(0.95) \approx$ [ ]

7. **[5 pt]** A runner on an elliptical track described by the equation $x^2 + \frac{y^2}{4} = 900$ runs at a constant speed v where x and y are measured in metres. Standing at the origin, you must rotate your head at $1/10$ radians per second to watch her as she crosses the finish line located on the x axis. How fast is she running?

$v =$ [ ]

C. Long-Answer Problems

In this section, to receive full credit, you must show your work and justify any claims unless otherwise stated.

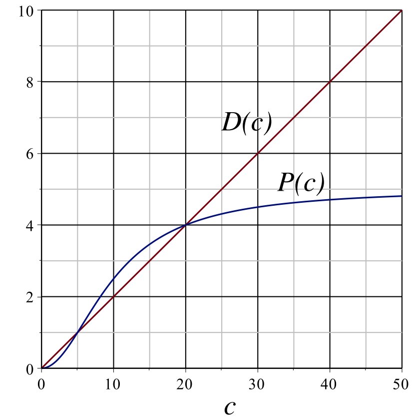
1. **[7 pt]** Use implicit differentiation to calculate the derivative of $\theta(x) = \operatorname{arccot}(x)$. Recall that $\cot(x) = \frac{\cos(x)}{\sin(x)}$.

2. **[11 pt]** The graphs to the right show two functions,

$$P(c) = P_m \frac{c^2}{k^2 + c^2}, \quad D(c) = rc$$

where $P_m = 5$, $k = 10$ and $r = 1/5$. Suppose that $c(t)$ is the concentration of a substance involved in a chemical reaction and satisfies the equation

$$\frac{dc}{dt} = P(c) - D(c).$$



(a) Calculate the steady states of the differential equation (show your work).

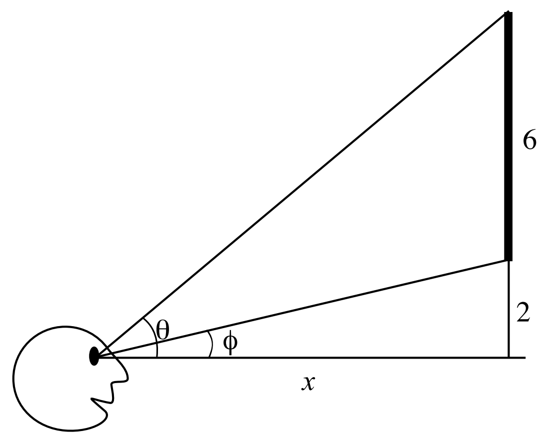
(b) Determine the stability of each steady state. Explain how you arrived at your conclusions either in words or using a diagram.

(c) If the concentration is initially $c(0) = 8$, what concentration does $c(t)$ eventually approach?

(d) Suppose that P_m and k are as given above. For what value of r does the differential equation for $c(t)$ have only two steady states?

3. **[7 pt]** At the outdoor summer movie at Stanley Park, the bottom of the screen is 2 meters above your eye level, and the screen is 6 meters tall. At what distance x from the base of the screen is the visual angle occupied by the screen as large as possible?

HINT: There are several possible approaches to this problem but one approach is to define θ as the angle to the top of the screen and ϕ as the angle to the bottom of the screen and maximize their difference.



This page may be used for rough work. It will not be marked.

Formula list

$$a^2 = b^2 + c^2 - 2bc\cos(\theta)$$

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

$$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \quad \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \quad \tan\left(\frac{\pi}{4}\right) = 1$$