Math 102- Final examination

University of British Columbia

December 11, 2013, 3:30 pm to 6:00 pm

Name (print):

ID number:

Section number:

This exam is "closed book". Calculators or other electronic aids are not allowed.

A		16
B		24
C.1		6
C.2		8
C.3		10
C.4		6
Total		70

A. Multiple choice questions

Enter your choice for each multiple choice question **in the box at the bottom of the page**. There are two pages at the end of the exam that can be used for rough work. No partial marks will be given for this section.

A.1 Let $f(x) = \sin(x) + ax^2$. Which of the following conditions describes all values of a for which f HAS inflection points?

(a) $|a| > 1/2$ (b) $a > 1/2$ (c) $a \geq 1/2$ (d) $|a| < 1/2$ (e) $|a| \leq 1/2$

A.2 Which of the following gives a value of $\cos\left(\arcsin\left(\frac{\sqrt{9-x^2}}{3}\right)\right)$?

(a) $3/x$ (b) $x/3$ (c) $\frac{\sqrt{9-x^2}}{x}$ (d) $\frac{x}{\sqrt{9-x^2}}$ (e) $\frac{3}{\sqrt{9-x^2}}$

A.3 Consider the differential equation and initial condition

$$\frac{dy}{dt} = 2 - y^2, \quad y(0) = 1.$$

Use Euler's method with one step of size $\Delta t = 0.1$ to approximate the value of the solution at time $t = 0.1$.

(a) $y(0.1) = 2$, (b) $y(0.1) = 2.1$, (c) $y(0.1) = 2.2$, (d) $y(0.1) = 1.2$, (e) $y(0.1) = 1.1$.

Answers:

Q A.1 [2 pts.]	Q A.2 [2 pts.]	Q A.3 [2 pts.]

A.4 Shown in the figure below is a function and its tangent line at $x = x_0$. The tangent line intersects the x axis at the point $x = x_1$. The coordinate of the point x_1 is

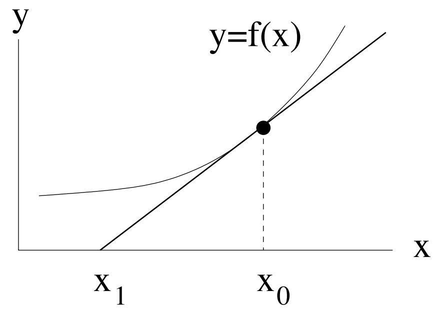
(a) $x_1 = x_0 + \frac{f(x_0)}{f'(x_0)}$,

(b) $x_1 = x_0 - f'(x_0)(x - x_0)$,

(c) $x_1 = x_0 - \frac{f'(x_1)}{f(x_1)}$,

(d) $x_1 = x_0 + \frac{f'(x_1)}{f(x_1)}$,

(e) $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$



A.5 Consider the function $y = \cos(x)$ and the tangent line to this function at the point $x = \pi/2$. Using that tangent line as a linear approximation of the function would lead to

(a) overestimating the value of the actual function for any nearby x.
(b) underestimating the value of the actual function for any nearby x.
(c) overestimating the function when $x > \pi/2$ and underestimating the function when $x < \pi/2$.
(d) overestimating the function when $x < \pi/2$ and underestimating the function when $x > \pi/2$.
(e) overestimating the function when $x < 0$ and underestimating the function when $x > 0$.

A.6 The slope field (direction field) shown in the figure below corresponds to which differential equation?

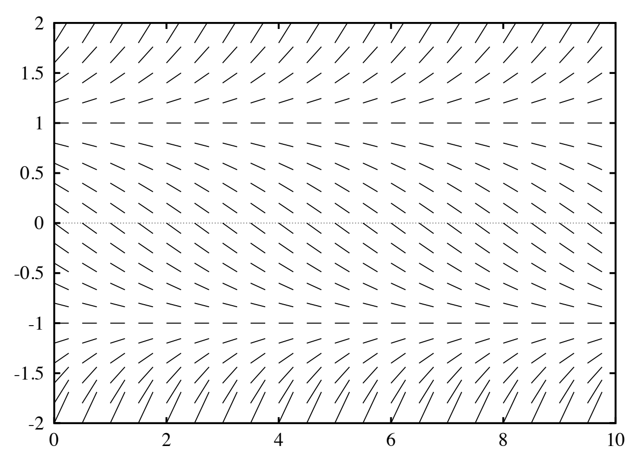
(a) $\frac{dy}{dt} = y(y + 1)$

(b) $\frac{dy}{dt} = (y - 1)(y + 1)$

(c) $\frac{dy}{dt} = -(y - 1)(y + 1)$

(d) $\frac{dy}{dt} = y(y - 1)$

(e) $\frac{dy}{dt} = -y(y + 1)$



Answers:

Q A.4 [2 pts.]	Q A.5 [2 pts.]	Q A.6 [2 pts.]

A.7 The number of sunspots (solar storms on the sun) varies with a period of roughly 11 years reaching a high of 120 and a low of 0 sunspots detected. A peak of 120 sunspots was detected in the year 2000. Which of the following trigonometric functions could be used to approximate this cycle?

(a) $N = 60 + 120\sin\left(\frac{2\pi}{11}(t-2000) + \frac{\pi}{2}\right)$, (b) $N = 60 + 60\sin\left(\frac{11}{2\pi}(t+2000)\right)$,

(c) $N = 60 + 60\cos\left(\frac{11}{2\pi}(t+2000)\right)$, (d) $N = 60 + 60\sin\left(\frac{2\pi}{11}(t-2000)\right)$,

(e) $N = 60 + 60\cos\left(\frac{2\pi}{11}(t-2000)\right)$.

A.8 Given the differential equation and initial condition

$$\frac{dy}{dt} = y^2(y-a), \quad y(0) = 2a$$

where $a > 0$ is a constant, which of the following is true?

(a) $\lim_{t\to\infty} y(t) = 0$.

(b) $\lim_{t\to\infty} y(t) = \infty$.

(c) $\lim_{t\to\infty} y(t) = a$.

(d) $\lim_{t\to\infty} y(t) = 2a$.

(e) None of the above.

Answers:

Q A.7 [2 pts.]	Q A.8 [2 pts.]

B. Short-answer problems

Show your work.

B.1 [3 pts.] Use implicit differentiation to find dy/dx where $y(x)$ is defined implicitly by $\tan(y) = x$. Your answer should not include any trig functions. Show your work.

$$\frac{dy}{dx} = \boxed{}$$

B.2 [3 pts.] A triangle has two sides with fixed lengths 3 cm and 4 cm respectively. The angle between them changes at a constant rate of 1 radian/sec. How quickly is the length of the third side (S) changing when the angle between the sides of fixed length is $\pi/2$? [See formula list on the last page of the exam.]

$$\frac{dS}{dt} = \boxed{}$$

B.3 [3 pts.] List the x coordinates of all local minima and maxima of $f(x) = x^5 - \frac{5}{4}x^4$.

Minima: $\boxed{}$

Maxima: $\boxed{}$

B.4 [3 pts.] What function satisfies $y'' + 4y = 0$ with initial condition $y(0) = 1$?

$y(t) =$ ☐

B.5 [3 pts.] Compute the limit $L = \lim_{h \to 0} \frac{\sqrt{3+h} - \sqrt{3}}{h}$.

$L =$ ☐

B.6 [3 pts.] Let

$$f(x) = \begin{cases} x^2 & x < 1 \\ a - x & x \geq 1 \end{cases}$$

where a is a constant. What value of a ensures that f is continuous for all x?

$a =$ ☐

B.7 [3 pts.] Initially, a patient has 1000 copies of the HIV virus. If the smallest detectable viral load is 350,000 virus particles, how long (in days) will it take until the HIV infection is detectable? Assume that the number of virus particles y grows according to the equation

$$\frac{dy}{dt} = 0.05y$$

where t is time in days. Leave your answer in terms of logarithms.

$t_{detect} = \boxed{}$

B.8 [3 pts] The velocity of a crawling cell has been modelled as

$$v(x) = \frac{1}{ax + e^{1/x}}$$

where $x > 0$ is the number of actin filaments driving the motion and $a > 0$ is a constant. Determine the following.

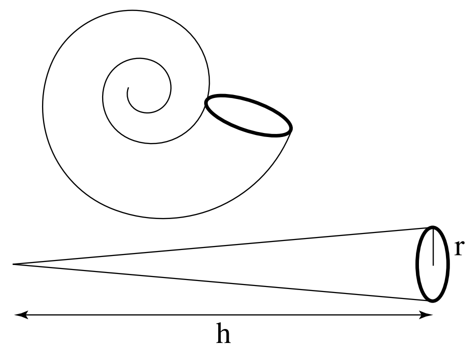
(i) $\lim_{x \to 0^+} e^{1/x} = \boxed{}$, (ii) $\lim_{x \to \infty} e^{1/x} = \boxed{}$

(iii) $\lim_{x \to 0^+} v(x) = \boxed{}$, (iv) $\lim_{x \to \infty} v(x) = \boxed{}$

C. Long-answer problems

C.1 [6 pts.] A mollusc grows inside a shell (top) and produces new shell material at its opening (thick curve). The volume of the shell can be approximated as that of a cone of base radius r and height h. Given that the height of the cone grows at a constant rate of 0.1 cm/year, at what rate will the volume of the cone be changing when $h = 10$ and $r = 1$ cm? Leave your answer in terms of π.

[See formula list on the last page of the exam.]



C.2 [8 pts.] The population of fish in a particular lake is given by the function $F(t)$ where F is measured in number of fish and t is measured in days. A company that manages fish stocks is hired to restock the lake, adding fish at a constant rate. Only N fishers are allowed to fish in the lake at a time. A simple model for this scenario is given by the equation

$$\frac{dF}{dt} = I - \alpha N F \tag{1}$$

where I and α are constant and two cases for N are considered.

(a) Explain in words what the constant I represents.

(b) Explain in words what the constant α represents.

(c) Case I: Suppose N is a constant. What is the steady state number of fish (F_*) in the lake? If the lake has no fish in it initially, at what time (t_*) does the population size reach half its steady state value $F_*/2$? (You do not have to show a derivation of the solution $F(t)$ of equation (1) for full points - simply stating it is sufficient - but you must show the rest of the calculation.)

$F_* =$ ☐

$t_* =$ ☐

Continued on next page...

(d) Case II: The agency that administers fishing permits decides that the number of fishers should be proportional to the number of fish currently in the lake, $N = F/3$ in equation (1). What is the population size F_* to which $F(t)$ approaches as $t \to \infty$ in this scenario?

$F_* =$ ☐

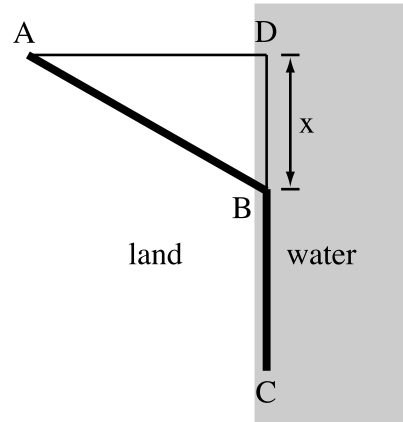
C.3 [10 pts.] Consider the function

$$f(x) = x^2 e^{-x}$$

defined on the whole real line.

Find all zeros of f, f' and f'' and determine all local minima, maxima and inflection points of f. Sketch the graph of f. You may use the fact that $\lim_{x \to \infty} x^2 e^{-x} = 0$.

C.4 [6 pts.] Shown in the figure below is the view from above of the path taken by a penguin from point A to a feeding area on the shore at point C. The penguin must choose the point B toward which it starts walking. It takes twice as much energy per unit distance for a penguin to walk over land (AB) as to swim through water (BC). The distance AD is 300 m and the distance DC is 400 m. Calculate the value of the distance x (and hence the location of the point B - see figure) that minimizes the energy spent on the entire trip.



$x_{min} =$ [ ]

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Formula list

$$a^2 = b^2 + c^2 - 2bc\cos(\theta)$$

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

$$V_{cone} = \frac{1}{3}\pi r^2 h$$