This examination has 13 pages of questions excluding this cover

The University of British Columbia

Final Examination, Saturday December 10$^{\text{th}}$

Mathematics 102: Differential Calculus with Applications to Life Sciences

101 (Kim), 102 (Hiller), 104 (Hauert), 105 (Chen), 106 (Coombs)

Closed book examination Time: 150 minutes

Last Name: ________________ **First Name:** ________________

Student Number: ________________ **Section:** circle above

Rules governing examinations:

1. No books, notes, electronic devices or any papers are allowed. To do your scratch work, use the back of the examination booklet. Additional paper is available upon request.
2. You should be prepared to produce your library/AMS card upon request.
3. No student shall be permitted to enter the examination room after 15 minutes and to leave within the first 15 minutes or less than 10 minutes before the completion of the examination.
4. You are not allowed to communicate with other students during the examination. Students may not purposely view other's written work nor purposefully expose his/her own work to the view of others or any imaging device.
5. At the end of the exam, you will put away all writing implements and calculators upon instruction. Students will continue to follow all of the above rules while the papers are being collected.
6. Students must follow all instructions provided by the invigilator.
7. Students are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions
8. Any deviation from these rules will be treated as an academic misconduct. The plea of accident or forgetfulness shall not be received.

I agree to follow the rules outlined above ________________

(signature)

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	15	15	10	11	10	9	10	10	10	100
Score:										

Show all your work and explain your reasonings clearly!

Express answers in terms of fractions or constants such as $\sqrt{e}$ or $\ln(4)$ rather than decimals.

1. (15 points) Short answer problems. A correct answer in the box gives full marks. For partial marks work needs to be shown.

   a. (6 points) Derivatives

      i. (4 points) The values of $f(x)$ are as given in the table.

x	1.00	1.50	2.00	2.50	3.00
$f(x)$	1.12	1.06	1.02	1.00	1.03

         a. Based on the above table, provide the best estimate for $f'(x)$ and fill in the second table.

x	1.25	1.75	2.25	2.75
$f'(x)$				

         b. Estimate $f''(2.0)$.

$f''(2.0) =$ ☐

      ii. (2 points) Find the derivative of $f(x) = \dfrac{x}{\ln(\frac{1}{x})}$

$\frac{df}{dx} =$ ☐

   b. (7 points) Slopes and tangents

      i. (2 points) Consider the curve $e^{2x} + y(1 - x) = 1$. Find $\frac{dy}{dx}$ at the point $x = 0$.

$\frac{dy}{dx} =$ ☐

ii. (2 points) Find the equation of the tangent line $y = mx + b$ to the graph of $f(x) = \cos(\pi x)$ at $x = 1/2$.

$m =$ ☐

$b =$ ☐

iii. (3 points) Let f^{-1} be the inverse function of $f(x)$. Assume $f(0) = 1$ and $f'(0) = 2$. Find the tangent line $y = mx + b$ to $f^{-1}(x)$ at 1.

$m =$ ☐

$b =$ ☐

c. (2 points) Consider the the function $f(x) = x^3 - x + 1$. Use one step of Newton's method to improve the initial guess $x_0 = -1$ for the zero (root) of $f(x)$.

$x_1 =$ ☐

2. (15 points) Short answer problems. A correct answer in the box gives full marks. For partial marks work needs to be shown.

   a. (6 points) Limits
   Note: If a limit does not exist, indicate whether it approaches $+\infty$ or $-\infty$.

   i. (2 points) Find the limit $A = \lim_{x\to\infty} \dfrac{2x^4 - 3x^3 + 5}{2x^3 - 5x^4}$.

   $A =$ [ ]

   ii. (2 points) Find the limit $B = \lim_{x\to 3} \dfrac{x^2 - 2x - 3}{x^2 - 3x}$.

   $B =$ [ ]

   iii. (2 points) Find the limit $C = \lim_{x\to\infty} \dfrac{\sin(e^{-x})}{e^{-x}}$.

   $C =$ [ ]

   b. (7 points) Differential equations

   i. (2 points) Consider the differential equation $\frac{dy}{dt} = y^3 - y^2 - 2y$.
   Find $\lim_{t\to\infty} y(t)$ for $y(0) = 1$.
   Hint: A sketch of the polynomial may be helpful.

   $\lim_{t\to\infty} y(t) =$ [ ]

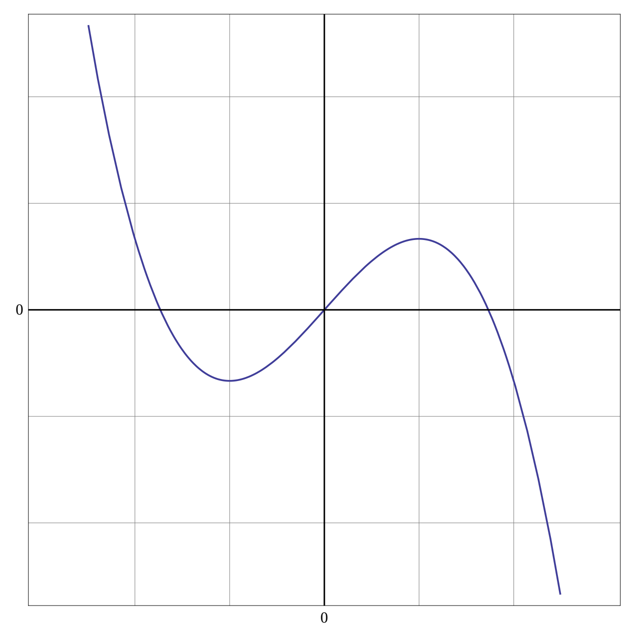
ii. (3 points) Circle the function that solves the following differential equation
$$\frac{dy}{dx} = 2xy.$$

(A) $y = xe^{2x}$, (B) $y = \frac{1}{2x}$, (C) $y = -e^{-x}$, (D) $y = -3e^{x^2}$, (E) $y = 3e^{-x^2}$

iii. (2 points) Using Euler's method, determine y_1, y_2, y_3 for $y_0 = 1$ and $\Delta t = \frac{1}{2}$
$$\frac{dy}{dt} = 3y - y^2.$$

$y_1 =$ ☐ $y_2 =$ ☐ $y_3 =$ ☐

c. (2 points) Consider the the function in the graph shown below. Draw a qualitatively accurate sketch of its derivative on top of it.



3. (10 points) Consider an ellipse given by the equation

$$\frac{x^2}{9} + y^2 = 1$$

and a point P in the interior with coordinates $(\frac{4}{9}, 0)$. Provide the x and y coordinates of the point(s) on the ellipse that are closest to P. Justify your answer.

4. (11 points) Consider the function $f(x) = \frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2$.

   i. (1 point) Determine $f(x)$ for $x \to \pm\infty$.

   $x \to \infty, f(x) \to$ [ ] $x \to -\infty, f(x) \to$ [ ]

   ii. (2 points) Find the x-values of all local minima and maxima of $f(x)$.

   iii. (2 points) Find the x-values of all inflection points of $f(x)$.
   Note: if minima, maxima or inflection points do not exist, say so.

   local min: [ ] local max: [ ]

   inflection points: [ ]

   iv. (2 points) Determine the interval(s) for which $f(x)$ is increasing and decreasing, respectively.

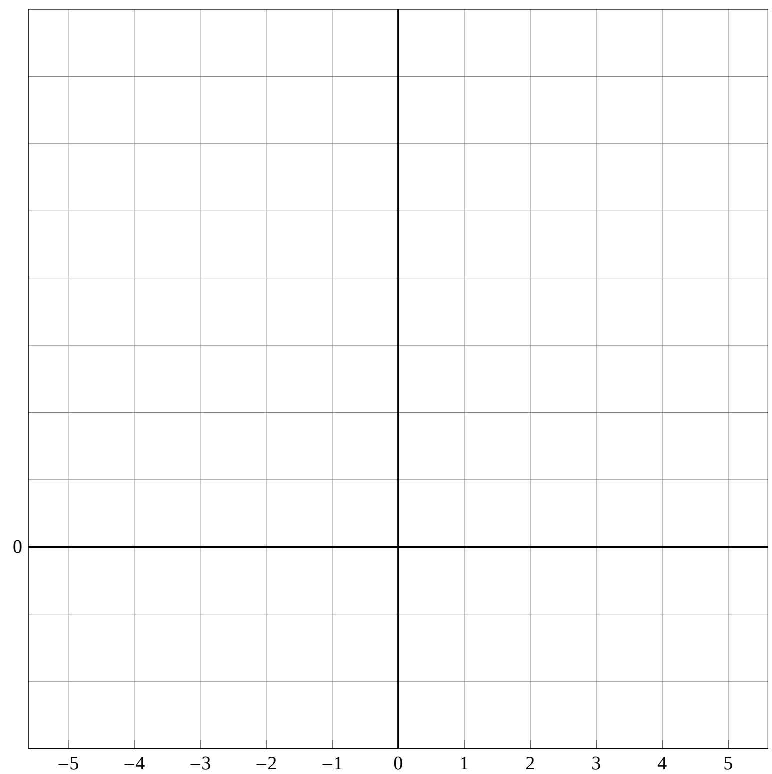
   increasing: [ ] decreasing: [ ]

Continued on next page.

v. (2 points) Determine the interval(s) for which $f(x)$ is concave up and concave down, respectively.

concave up: [ ] concave down: [ ]

vi. (2 points) Provide a qualitatively accurate sketch of $f(x)$. Make sure your graph reflects the information above.



5. (10 points) Find the absolute maximum and absolute minimum values of $f(x) = \sin(x) - \cos(x)$ on $[0, 2\pi]$.

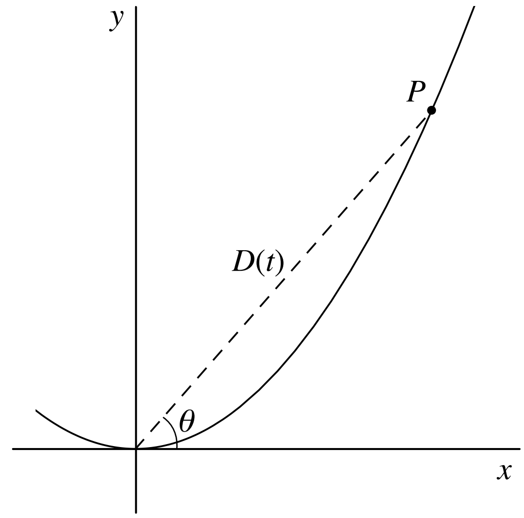
absolute minimum $(x, y) =$ [ ]

absolute maximum $(x, y) =$ [ ]

6. (9 points) Let $f(x) = \ln(x^3)$ and $g(x) = (\ln x)^3$. Find *all* values of x where the tangent lines of these two functions have the same slope (or show that no such values exist).

7. (10 points) A particle is moving in the x, y-plane along the curve $y = x^2$. As it passes through the point $P = (2, 4)$, its y coordinate changes at a rate of 8 units/sec.

   i. (3 points) Find $\frac{dx}{dt}$ at the instant the particle is at the point P.



$\frac{dx}{dt} =$ ☐

   ii. (3 points) What is the rate of change of the particle's distance $D(t)$ to the origin at this instant?

$\frac{dD}{dt} =$ ☐

   iii. (4 points) What is the rate of change of the angle θ between the positive x-axis and the line connecting the origin and the particle?

$\frac{d\theta}{dt} =$ ☐

8. (10 points) You are driving down the highway when you see a sleeping moose. You apply the brakes and carefully stop your car 20m away from the animal. While you are looking for your camera the moose wakes up. It instantly charges toward your car at a constant speed of 8m/s. One second later, you start backing away from the moose at a constant acceleration of 2m/s^2.

   i. (4 points) Write down a function $d(t)$ that is the distance from your car to the moose where $t = 0$ indicates the moment when you start backing away.

   $d(t) =$ [ ]

   ii. (3 points) At what time T does the moose hit your car?

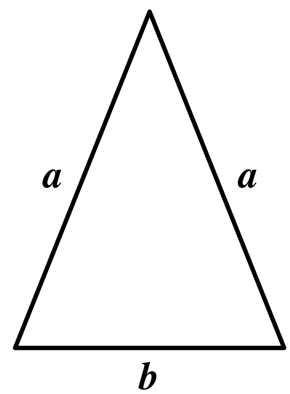
   $T =$ [ ]

   iii. (3 points) A few days later you are shopping for a new car. You would like to purchase a faster car to avoid moose accidents. What is the minimum acceleration a that your car would have needed to avoid being caught by the moose in the previous scenario?

   $a =$ [ ]

9. (10 points) Consider the isosceles triangle given in the following figure, where a and b indicate the side lengths. Assume that the triangle has a circumference of 2. Find the lengths a and b for which the area A of the triangle is maximized. You must also check that you found a maximum and your solution must include that check.

   Hint: Heron's formula may be useful, which states that $A^2 = s(s-x)(s-y)(s-z)$, where A is the area of the triangle, x, y, z its side lengths and $s = \frac{x+y+z}{2}$.



$a =$ ☐ $b =$ ☐

Useful Formulæ

Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Trigonometric identities

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta; \qquad \text{for } \alpha=\beta: \quad \sin(2\alpha) = 2\sin\alpha\cos\alpha$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta; \qquad \text{for } \alpha=\beta: \quad \cos(2\alpha) = 2\cos^2\alpha - 1$$
$$\sin^2\alpha + \cos^2\alpha = 1$$
$$\tan^2\alpha + 1 = \sec^2\alpha = \frac{1}{\cos^2\alpha}$$
$$\sin\alpha = \sin(\pi-\alpha); \qquad \cos\alpha = \cos(2\pi-\alpha)$$

Some useful trigonometric values

$$\sin(0)=0, \quad \sin(\frac{\pi}{6}) = \frac{1}{2}, \quad \sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}, \quad \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}, \quad \sin(\frac{\pi}{2}) = 1, \quad \sin(\pi) = 0$$
$$\cos(0)=1, \quad \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}, \quad \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}, \quad \cos(\frac{\pi}{3}) = \frac{1}{2}, \quad \cos(\frac{\pi}{2}) = 0, \quad \cos(\pi) = -1$$