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The University of British Columbia
Sessional Examinations - April 2012

Mathematics 101
Integral Calculus with Applications to Physical Sciences and Engineering

Closed book examination Time: 2.5 hours

Last Name: ______________ **First Name:** ______________

Student Number: ______________ **Instructor's Name:** ______________

Signature: ______________ **Section Number:** ______________

Rules governing examinations

1. Each candidate must be prepared to produce, upon request, a UBC-card for identification.
2. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
3. No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.
4. Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
   (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
   (b) Speaking or communicating with other candidates.
   (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
5. Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material form the examination room without permission of the invigilator.
6. Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		30
2		10
3		20
4		12
5		8
6		8
7		6
8		6
Total		100

Marks

[30] **1. Short-Answer Questions.** Put your answer in the box provided. **Simplify your answer as much as possible.** Full marks will be awarded for a correct answer placed in the box. **Show your work,** for part marks. Each question is worth 3 marks, but not all questions are of equal difficulty.

(a) Evaluate $\displaystyle\int_1^2 \frac{x^2+2}{x^2}\,dx$.

Answer

(b) Evaluate $\displaystyle\int_0^{\pi/2} \frac{\cos x}{1+\sin x}\,dx$. Remember to simplify your answer completely.

Answer

(c) Evaluate $\displaystyle\int_{-2012}^{2012} x^{1/3}\cos x\,dx$.

Answer

Continued on page 3

(d) Let k be a positive constant. Find the average value of the function $f(x) = \sin(kx)$ on the interval $[0, \pi/k]$.

Answer

(e) A function $f(x)$ is always positive and satisfies $f'(x) = xf(x)$ for all x and also $f(0) = e$. Find this function.

Answer

(f) Find the y-coordinate of the centroid of the region bounded by the curves $y = 1$, $y = -e^x$, $x = 0$, and $x = 1$. You may use the fact that the area of this region equals e.

Answer

Continued on page 4

(g) Let $R_n = \sum_{i=1}^{n} \frac{ie^{i/n}}{n^2}$. Express $\lim_{n\to\infty} R_n$ as a definite integral. *Do not evaluate this integral.*

Answer

(h) Express $2.656565\cdots$ as a rational number, i.e. in the form p/q where p and q are integers.

Answer

(i) Evaluate $\lim_{x\to 0} \frac{\sin x - x + x^3/6}{\sin(x^5)}$.

Answer

(j) Using a Maclaurin series, the number $a = 1/5 - 1/7 + 1/18$ is found to be an approximation for $I = \int_0^1 x^4 e^{-x^2}\, dx$. Give the best upper bound you can for $|I - a|$.

Answer

Continued on page 5

Full-Solution Problems. In questions 2–8, justify your answers and **show all your work**. If a box is provided, write your final answer there. **Unless otherwise indicated, simplification of numerical answers is required in these questions.**

[10] **2.** Let R be the bounded region that lies between the curve $y = 4 - (x-1)^2$ and the $y = x + 1$.

(a) [6] Sketch R and find its area.

(b) [4] Write down a definite integral giving the volume of the region obtained by rotating R about the line $y = 5$. *Do not evaluate this integral.*

Answer

Continued on page 6

[20] **3.** (a) [5] Evaluate $\int \frac{\ln x}{x^{101}}\ dx$

Answer

(b) [5] Evaluate $\int_0^3 (x+1)\sqrt{9-x^2}\ dx$

Answer

Continued on page 7

(c) [5] Evaluate $\int \frac{4x+8}{(x-2)(x^2+4)}\ dx$

Answer

(d) [5] Determine, with explanation, whether the improper integral $\int_{-\infty}^{\infty} \frac{x}{x^2+1}\ dx$ converges or diverges.

Continued on page 8

[12] **4.** (a) [4] Determine, with explanation, whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 - \sin n}{n^6 + n^2}$ converges or diverges.

(b) [4] Determine, with explanation, whether the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(n^2+1)(n!)^2}$ converges absolutely, converges conditionally, or diverges.

(c) [4] Determine, with explanation, whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^{101}}$ converges absolutely, converges conditionally, or diverges.

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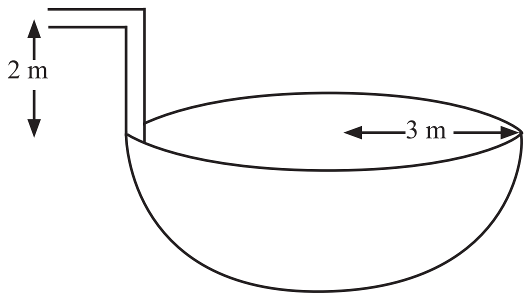
[8] **5.** Find, with explanation, the radius of convergence and the interval of convergence of the power series

$$\sum_{n=0}^{\infty}(-1)^n \frac{(x-1)^n}{2^n(n+2)}$$

Answer

Continued on page 10

[8] **6.** A tank in the shape of a hemispherical bowl of radius 3 m, with an outlet that rises 2 m above its top (see the diagram below), is full of water. Using the fact that the density of water is 1000 kg/m^3, find the work (in Joules) required to pump all the water out of the outlet. You may use the value $g = 9.8$ m/s^2 for the acceleration due to gravity. *You do not need to simplify your answer, but you must completely evaluate any integral(s) that arise.*



Answer

Continued on page 11

[6] **7.** Let $I = \int_1^2 (1/x)\,dx$.

(a) [2] Write down the trapezoidal approximation T_4 for I. *You do not need to simplify your answer.*

(b) [2] Write down the Simpson's approximation S_4 for I. *You do not need to simplify your answer.*

(c) [2] Without computing I, find an upper bound for $|I - S_4|$. You may use the fact that if $|f^{(4)}(x)| \le K$ on the interval $[a, b]$, then the error in using S_n to approximate $\int_a^b f(x)\,dx$ has absolute value less than or equal to $K(b-a)^5/180n^4$.

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[6] **8.** (a) [3] Evaluate $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^n}$.

Answer

(b) [3] Prove that for any positive real number a, $\int_0^{\pi/2} e^{-a\sin x}\, dx \le \frac{\pi}{2a}$. Hint: First find a simple function $f(x)$ such that $\sin x \ge f(x)$ for $0 \le x \le \pi/2$.

The End