Be sure that this examination has 11 pages including this cover

The University of British Columbia

Sessional Examinations - April 2008

Mathematics 101

Integral Calculus with Applications to Physical Sciences and Engineering

Closed book examination | Time: 2.5 hours

Last Name: ________________ **First Name:** ________________

Student Number:________________ **Instructor's Name:** ________________

Signature:________________ **Section Number:** ________________

Rules governing examinations

1. Each candidate should be prepared to produce his or her library/AMS card upon request.
2. Read and observe the following rules:

No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.

Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

(a) Making use of any books, papers or memoranda, other than those authorized by the examiners.

(b) Speaking or communicating with other candidates.

(c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

3. Smoking is not permitted during examinations.

1		21
2		20
3		20
4		12
5		10
6		10
7		7
Total		100

Marks

[21] **1.** **Short-Answer Questions.** Put your answer in the box provided but show your work also. Each question is worth 3 marks, but not all questions are of equal difficulty. Full marks will be given for correct answers placed in the box, but at most 1 mark will be given for incorrect answers. Unless otherwise stated, simplify your answer as much as possible.

(a) Evaluate $\displaystyle\int \frac{x^3 - 2x}{\sqrt{x}}\, dx$.

Answer

(b) Evaluate $\displaystyle\int_0^{\pi} (4\sin\theta - 3\cos\theta)\, d\theta$. You must simplify your answer *completely.*

Answer

(c) Express $\displaystyle\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{1}{1+(i/n)^2}$ as a definite integral. *Do not* evaluate this integral.

Answer

Continued on page 3

(d) Write down the Trapezoidal Rule approximation T_3 for $\int_1^4 x\cos(\pi/x)\,dx$. Leave your answer expressed as a sum involving cosines.

Answer

(e) Let $f(x) = kx^2(1-x)$ if $0 \le x \le 1$ and $f(x) = 0$ if $x < 0$ or $x > 1$. For what value of the positive constant k is $f(x)$ a probability density function?

Answer

(f) Find the first three nonzero terms in the power-series representation in powers of x (i.e. the Maclaurin series) for $\int_0^x \frac{t}{1-t^8}\,dt$.

Answer

(g) Let $f(x) = \int_x^{x^3} \sqrt{t}\sin t\,dt$. Find $f'(x)$.

Answer

Continued on page 4

Full-Solution Problems. In questions 2–7, justify your answers and **show all your work**. If a box is provided, write your final answer there. Unless otherwise indicated, simplification of answers is not required.

[20] **2.** (a) [5] Sketch the bounded region that lies between the curves $y = 2x^2$ and $y = 4 + x^2$, and find its area. (Place only your answer for the area in the answer box.)

Answer

(b) [5] Let R be the unbounded region that lies under the curve $y = 1/x^p$, above the x-axis, and to the right of the vertical line $x = 1$. For what values of the constant p does the solid obtained by rotating R about the x-axis have *finite* volume?

Answer

Continued on page 5

(c) [5] Find the volume of the solid obtained by rotating the region bounded by the curves $y = 5$ and $y = x + (4/x)$ about the line $x = -1$.

Answer

(d) [5] A cable that weighs 2 lb/ft is used to lift 800 lb of coal up a mine shaft 500 ft deep. Find the *total* work done (including the work done in lifting the cable itself).

Answer

Continued on page 6

[20] **3.** Evaluate the following integrals.

(a) [5]

$$\int_1^2 \frac{e^{1/x}}{x^2}\, dx$$

Answer

(b) [5]

$$\int \cos\sqrt{x}\, dx$$

Hint: You will need to use a substitution combined with another method of integration.

Answer

Continued on page 7

(c) [5]

$$\int \frac{dx}{x(x^2+4)}$$

Answer

(d) [5]

$$\int \frac{dx}{\sqrt{x^2+16}}$$

Answer

Continued on page 8

[12] **4.** (a) [6] Solve the initial-value problem $2y'' + 5y' + 3y = 0$, $y(0) = 3$, $y'(0) = -4$.

Answer

(b) [6] Find the general solution of the differential equation $y'' - y' = \sin(2x)$.

Answer

Continued on page 9

[10] **5.** Let $I = \int_0^1 \cos(x^2)\ dx$.

(a) [6] Write down the first three nonzero terms obtained by using Maclaurin series to estimate I, and explain why the error in using this estimate is less than 0.001.

(b) [4] It can be shown that the 4th derivative of $\cos(x^2)$ has absolute value at most 60 on the interval $[0, 1]$. Using this bound, find the smallest positive integer n you can such that the Simpson's Rule approximation S_n for I has error less or equal to 0.001. You may use the fact that if $|f^{(4)}(t)| \leq K$ on the interval $[a, b]$, then the error in using S_n to approximate $\int_a^b f(x)\ dx$ has absolute value less than or equal to $K(b-a)^5/180n^4$.

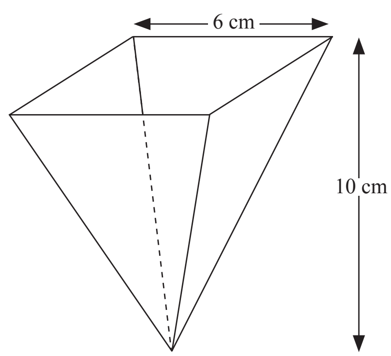
Answer

Continued on page 10

[10] **6.** A paper cup has the unusual shape depicted below. All of its horizontal cross sections are squares, with the top of the cup a square of side length 6 cm, and the cup has a height of 10 cm. The cup is initially full of Jolting Java, a potent coffee drink. The precious liquid is leaking from a small hole at the bottom of the cup. After 10 minutes, the height of the coffee above the bottom of the cup has decreased from 10 cm to 5 cm. After how many more minutes will the cup be completely empty? Assume the coffee drains according to Toricelli's Law, which is stated below. Here, y is the height of the top surface of the coffee above the bottom of the cup, $A(y)$ is the area of the horizontal cross-section of the cup at height y above the bottom, and k is a positive constant. (Also, assume that no coffee is drunk or lost to evaporation.)

$$\text{Toricelli's Law :} \quad A(y)\frac{dy}{dt} = -k\sqrt{y}$$

Answer



Continued on page 11

[7] **7.** (a) [3] Show that the area of the region inside the ellipse $(x^2/a^2) + (y^2/b^2) = 1$, where a and b are positive constants, equals πab.

(b) [4] Let E be the ellipse $x^2 + k^2y^2 = 1$, where k is a constant and $k \geq 1$. Let S be the region inside the circle $x^2 + y^2 = 1$, outside E, and above the x-axis. Find all values of k such that the centroid (centre of mass) of S lies inside S (i.e. outside E). You may use the result of part (a) above.

Answer

The End