Be sure that this examination has 12 pages including this cover

The University of British Columbia
Sessional Examinations - April 2005

Mathematics 101
Integral Calculus

Closed book examination Time: 2.5 hours

Print Name ____________________ **Signature** ____________________

Student Number ____________________ **Instructor's Name** ____________________

Section Number ____________________

Special Instructions:

No calculators, cell phones, notes, or books of any kind are allowed. Show all calculations for your solutions. If you need more space than is provided, use the back of the previous page.
Where boxes are provided for answers, put your final answers in them.

Rules governing examinations

1. Each candidate should be prepared to produce his or her library/AMS card upon request.
2. Read and observe the following rules:

No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.

Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

(a) Making use of any books, papers or memoranda, other than those authorized by the examiners.

(b) Speaking or communicating with other candidates.

(c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

3. Smoking is not permitted during examinations.

1		21
2		15
3		16
4		10
5		12
6		10
7		10
8		6
Total		100

Marks

[21] **1.** **Short-Answer Questions.** Put your answer in the box provided but show your work also. Each question is worth 3 marks, but not all questions are of equal difficulty. Full marks will be given for correct answers placed in the box, but at most one mark will be given for incorrect answers. Simplify your answers as much as possible.

(a) Evaluate

$$\int (x^2 + e^{2x})\, dx$$

Answer

(b) Find the average value of $\sin x$ on the interval $[0, \pi]$.

Answer

(c) If

$$F(x) = \int_0^{x^2} f(t)\, dt,$$

where f is a function satisfying $f(4) = 1$, compute $F'(2)$.

Answer

Continued on page 3

(d) Find the general solution $y = y(x)$ of the differential equation

$$y'' - 2y' + y = 0$$

Answer

(e) Find the general solution $y = y(x)$ of the differential equation

$$y'' - 2y' + y = x$$

Answer

(f) Evaluate

$$\int_0^\infty \frac{dx}{(x+1)^3}$$

Answer

(g) A continuous random variable X is exponentially distributed with a mean of 4. Find the probability that $X \geq 8$.

Answer

Continued on page 4

Full-Solution Problems. In questions 2–8, justify your answers and **show all your work**. If a box is provided, write your final answer there. Unless otherwise indicated, simplification of answers is not required.

[15] **2.** Let R be the finite region bounded above by the curve $y = 4 - x^2$ and below by $y = 2 - x$.

(a) [4] Carefully sketch R and find its area explicitly (place your answer for the area only in the answer box).

Answer

(b) [3] Express the volume of the solid obtained by rotating R about the x-axis as a definite integral. You do not need to simplify or evaluate this integral.

Answer

Continued on page 5

(c) [4] Express the volume of the solid obtained by rotating R about the vertical line $x = 2$ as a definite integral. You do not need to simplify or evaluate this integral.

Answer

(d) [4] Express the length of the upper curve that bounds R as a definite integral, and using an appropriate substitution express your answer as an integral involving trigonometric functions. You do not need to evaluate this trigonometric integral.

Answer

Continued on page 6

[16] **3.** Evaluate the following integrals.

(a) [5]

$$\int \frac{4x+4}{x(x+1)^2}\ dx$$

Answer

(b) [4]

$$\int (x+1)\ln x\ dx$$

Answer

Continued on page 7

(c) [7]

$$\int \frac{dx}{(5-4x-x^2)^{3/2}}$$

Answer

Continued on page 8

[10] **4.** A mass attached to a spring satisfies the differential equation

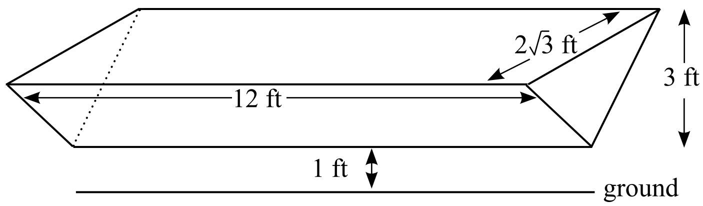
$$x'' + 4x' + 3x = 60\cos 3t,$$

where $x = x(t)$ is the position of the mass at time t. Find $x(t)$, given that $x(0) = x'(0) = 0$ (i.e. solve the initial-value problem).

Answer

Continued on page 9

[12] **5.** A water tank with depth 3 feet is in the shape of the trough depicted below. The bottom edge of the tank is 1 foot above the ground, and the ends of the tank are equilateral triangles of side length $2\sqrt{3}$ feet; the top of the tank is a rectangle of length 12 feet and width $2\sqrt{3}$ feet.



Answer

(a) [6] The tank is filled with water pumped up from ground level. How much work is done in filling the tank? Express your answer in foot-pounds, and use the value 62 lb/ft^3 for the density of water. [Note that in the Imperial System, pounds are a unit of *force*.] (Do not simplify your answer, and place your answer in the box above.)

(b) [6] Sometime after it is filled, the tank develops a small hole at its bottom. The tank then drains according to Toricelli's Law:

$$A(y)\frac{dy}{dt} = -k\sqrt{y},$$

where y is the height of the water in the tank above its bottom, $A(y)$ is the area of the horizontal cross-section of the tank at height y, and k is a positive constant. If the water level in the tank drops to 1 foot after 2 hours, how long will it take for the tank to empty? (Do not simplify your answer.)

Answer

Continued on page 10

[10] **6.** A tree trunk is 60 feet long and has circular cross sections. The diameters in feet measured at 10-foot intervals starting at the bottom of the tree are 10, 8, 7, 6, 5, 4, and 3. The volume of the tree, in cubic feet, is computed using the formula

$$V = \int_0^{60} A(y)\ dy,$$

where $A(y)$ is the cross-sectional area of the trunk, in square feet, y feet above the bottom of the tree.

(a) [6] Using a Simpson's Rule approximation for V, give an estimate for the volume of wood in the tree trunk. **Do not simplify your answer.**

(b) [4] It is known that the fourth derivative of $A(y)$ has values that lie between 3 and $30,000$. How many measurements of the tree's diameter (equally spaced along the tree trunk) would be needed in order to estimate its volume using Simpson's Rule to an accuracy of 1 cubic foot? You may use the fact that the error made in approximating a definite integral $\int_a^b f(x)\ dx$ by the corresponding Simpson's Rule approximation S_n is at most $(M(b-a)^5)/(180n^4)$, where $|f^{(4)}(x)| \leq M$ on the interval $[a, b]$. You must give your answer as an explicit integer for full marks.

Answer

Continued on page 11

[10] **7.** The length of time in minutes it takes students to solve a certain mathematics problem (on probability) is a continuous random variable whose probability density function is

$$f(x) = \begin{cases} k\cos(x/10)\sin^2(x/10) & \text{if } 0 \le x \le 5\pi \\ 0 & \text{otherwise} \end{cases}$$

(a) [3] Find the value of the positive constant k.

Answer

(b) [3] What is the probability that a student will take less than $5\pi/2$ minutes to complete the problem?

Answer

(c) [4] Compute the mean length of time required to solve the problem.

Answer

Continued on page 12

[6] **8.** An unknown continuous function $f(x)$ satisfies $f(0) = 0,\ f(4) = 8$, and

$$2x \leq f(x) \leq 6x - x^2$$

for $0 \leq x \leq 4$. Also, $f(x)$ is *nondecreasing* on this interval, i.e. it satisfies $f(c) \leq f(d)$ for all real numbers c and d with $0 \leq c \leq d \leq 4$. Let I be the value of definite integral $\int_0^4 f(x)\,dx$.

(a) [2] Let L_{100} be the underestimate for I obtained by using a Riemann sum with equal-length subintervals and $x_i^* = x_{i-1}$ (i.e. using the left endpoints of the subintervals), and R_{100} be the overestimate obtained by using $x_i^* = x_i$ (i.e. the right endpoints). Compute a numerical value for $R_{100} - L_{100}$.

Answer

(b) [4] Find the smallest and largest possible values for I. Give explicit functions $f(x)$, satisfying the conditions above, that yield these smallest and largest values.

Answer

The End