

Final Examination — December 16th 2015

Duration: 2.5 hours

This test has 12 questions on 17 pages, for a total of 100 points.

- Read all the questions carefully before starting to work.
- Q1-Q3 are multiple choice questions; write your answer in the box provided.
- Q4-Q6 are short-answer questions; put your answer in the boxes provided.
- All other questions are long-answer; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed:** documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: _____ Last Name: _____

Student-No: _____ Section: _____

Signature: _____

Question:	1	2	3	4	5	6	7	8	9	10	11	12	Total
Points:	4	4	8	8	12	12	6	7	7	14	8	10	100
Score:													

Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
 - (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
 - (iii) purposely viewing the written papers of other examination candidates;
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
 - (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
 - 6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
 - 7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
 - 8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Multiple choice questions:

4 marks

1. Each part is worth 1 mark. **Write your choice in the box provided.****ANSWERS HERE**

(i)	(ii)	(iii)	(iv)
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(i) Compute $\lim_{t \rightarrow 3} \frac{t-3}{(t+1)^2}$ A. -1 B. $-\frac{3}{16}$ C. 0 D. $\frac{1}{8}$ E. 3

F. does not exist

(ii) Compute $\lim_{x \rightarrow +\infty} \frac{x+1}{x^2+2x-8}$.A. 0 B. 1 C. 2 D. 4 E. $-\infty$ F. $+\infty$ (iii) Compute $\lim_{x \rightarrow 4^+} \frac{x}{(x-4)^2}$ A. -4 B. 0 C. 1 D. 3 E. $-\infty$ F. $+\infty$ (iv) Describe all points where the function $f(x) = \frac{e^x(x-1)}{x^2}$ is continuous.A. $(0, \infty)$ B. $(1, \infty)$ C. $(-\infty, 1) \cup (1, \infty)$ D. $(-\infty, 0) \cup (0, \infty)$ E. $(-\infty, 0) \cup (1, \infty)$ F. $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$

4 marks

2. Each part is worth 1 mark. **Write your choice in the box provided.****ANSWERS HERE**

(i)	(ii)	(iii)	(iv)
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(i) Find the derivative of $f(x) = \tan x$.

A. $\frac{1}{\cos(x)}$

B. $\frac{\cos(x)}{\sin(x)}$

C. $\frac{1}{\cos^2(x)}$

D. $\frac{\cos^2(x)}{\sin^2(x)}$

E. $-\frac{1}{\sin^2(x)}$

F. $-\frac{\cos(x)}{\sin^2(x)}$

(ii) Find the derivative of $f(x) = x \log x$. Remember $\log x = \log_e x = \ln x$.

A. $\frac{\log x}{x}$

B. $1 + \frac{\log x}{x}$

C. $x \log x$

D. $x - \log x$

E. $1 + \log x$

F. $x + \log x$

(iii) Find the derivative of $f(x) = \frac{x^2}{x+1}$.

A. $2x$

B. $\frac{x(x+2)}{x+1}$

C. $\frac{2x}{(x+1)^2}$

D. $\frac{x(x+2)}{(x+1)^2}$

E. $\frac{x^2(x+1)}{(x+2)^2}$

F. $\frac{x+1}{x(x+2)}$

(iv) Find the derivative of $f(x) = \sin(x^2)$.

A. $\sin(2x)$

B. $\cos(2x)$

C. $2x \sin(x^2)$

D. $2x \cos(x^2)$

E. $2x \cos(2x)$

F. $2 \sin(x) \cos(x)$

8 marks

3. Each part is worth 2 marks. **Write your choice in the box provided.****ANSWERS HERE**

(i)	(ii)	(iii)	(iv)
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(i) What is the equation of the line tangent to the function $f(x) = \sqrt{x}$ at the point $(4, 2)$?

A. $y = \frac{1}{4}x + 2$

D. $(y - 2) = \frac{1}{2}(x - 4)$

B. $(y - 2) = \frac{1}{4}(x - 4)$

E. $(y - 4) = \frac{1}{2}(x - 2)$

C. $(y - 2) = \frac{1}{2}(x - 2)$

F. $(y - 4) = 2(x - 2)$

(ii) Let $xy^2 + yx^2 = 2$. Find $\frac{dy}{dx}$ at the point $(1, 1)$.

A. -1

D. $1/3$

B. $-1/3$

E. 1

C. 0

F. 2

(iii) Let $f(x) = x^x$. What is $f'(x)$?

A. $x \cdot x^{x-1}$

D. $1 + \log x$

B. $x^x(1 + x^x)$

E. $x^x(1 + \log x)$

C. $x^x \log x$

F. xe^x

Remember $\log x = \log_e x = \ln x$.(iv) Let $T_3(x) = 24 + 6(x - 3) + 12(x - 3)^2 + 4(x - 3)^3$ be the third-degree Taylor polynomial for some function $f(x)$, expanded about $a = 3$. What is $f''(3)$?

A. 2

D. 6

B. 3

E. 12

C. 4

F. 24

Short-answer questions: Put your answer in the box provided. Full marks will be given for a correct answer placed in the box, while part marks may be given if workings are shown. Please simplify your answers.

2 marks

4. (a) Compute $\arcsin\left(\sin\left(\frac{3\pi}{5}\right)\right)$.

Answer:

2 marks

(b) Evaluate $\lim_{x \rightarrow 2} \frac{x^2 - 3x + 2}{x - 2}$.

Answer:

2 marks

(c) Evaluate $\lim_{x \rightarrow -\infty} \frac{x + \sqrt{4x^2 - x}}{6x}$.

Answer:

2 marks

(d) Find the domain of $f(x) = \frac{x^2 - 1}{\sqrt{x^2 - x - 6}}$.

Answer:

3 marks

5. (a) Consider the curve defined by $y^2 + 4xy - 2x^2 = 3$. Find the x -coordinates of all points on the curve where $\frac{dy}{dx} = 0$.

Answer:

3 marks

- (b) Let $f(x)$ be a function differentiable at $x = 3$ and let $g(x) = x \cdot f(x)$. The line tangent to the curve $y = f(x)$ at $x = 3$ has slope 2 while the line tangent to the curve $y = g(x)$ at $x = 3$ has slope 5. What is $f(3)$?

Answer:

3 marks

(c) Find the coordinates of the inflection point of the function $h(x) = x e^x$.

Answer:

3 marks

(d) Evaluate the limit $\lim_{x \rightarrow 0} \frac{\log(1+x) - \sin x}{x^2}$. Remember $\log x = \log_e x = \ln x$.

Answer:

3 marks

6. (a) A colony of bacteria doubles every 4 hours. If the colony has 2000 cells after 6 hours, how many cells were present initially? Simplify your answer.

Answer:

3 marks

- (b) Estimate $\sqrt{8}$ using a linear approximation.

Answer:

3 marks

(c) What are the critical points of the function $f(x) = e^{x^3 - 9x^2 + 15x - 1}$?

Answer:

3 marks

(d) Find a function $f(x)$ such that $f'(x) = \sin(x) + \frac{2}{\sqrt{x}}$ and such that $f(\pi) = 0$.

Answer:

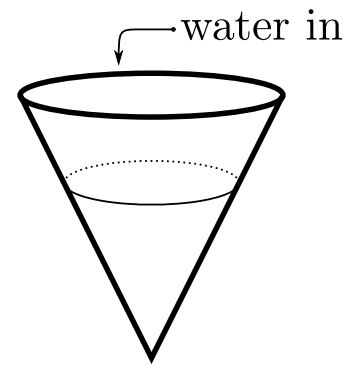
Full-solution problems: Justify your answers and **show all your work**. If a box is provided, write your final answer there. Unless otherwise indicated, **simplification of answers is not required in these questions**.

6 marks

7. Is the function

$$f(x) = \begin{cases} \sqrt{1+x^2} - 1 & \text{if } x \leq 0 \\ x^2 \cos(1/x) & \text{if } x > 0 \end{cases}$$

differentiable at $x = 0$? You must explain your answer using the definition of the derivative.



8. A tank of water in shape of an inverted circular cone of height 16m and diameter 20m at the top is being filled with water at a rate of $2\text{m}^3/\text{minute}$.

2 marks

- (a) Draw a vertical cross-section of the cone and label it with all the lengths that appear in your solution to part (b)

5 marks

- (b) Find the rate of change of the height of the water in the tank when the height of the water is 10m.

7 marks

9. A business wants to manufacture a huge closed box with a square base a square top and rectangular sides. The material used for the four upright sides of the box costs \$2 per square metre, and the material used for the base and top of the box costs \$8 per square metre.

Find the dimensions of the box with lowest possible cost that has volume 32m^3 .

10. Let $f(x) = (x + 2)(x + 1)e^{-x}$.

1 mark

(a) Find the domain of the function.

Answer:

2 marks

(b) Find the x -intercepts and y -intercepts of the function.

Answer:

2 marks

(c) Find the horizontal asymptotes of the function (if they exist).

Answer:

1 mark

(d) Find the vertical asymptotes of the function (if they exist).

Answer:

1 mark

(e) Find $f'(x)$. You must simplify your answer.

Answer:

2 marks

(f) Find all critical points and singular points of $f(x)$.

Answer:

3 marks

(g) Find the intervals on which $f(x)$ is increasing. Find the intervals on which $f(x)$ is decreasing.

2 marks

(h) The second derivative is $f''(x) = (x - 2)(x + 1)e^{-x}$. Find the x -coordinates of any points of inflection. You must explain your answer.

- 4 marks 11. (a) If the second degree Maclaurin polynomial for $g(x)$ is $T_2(x) = 5 - x/3 + 2x^2$ find the second degree Maclaurin polynomial for $h(x) = e^x \cdot g(x)$.

- 4 marks (b) If $f^{(3)}(x) = \frac{x \sin(x) + x^2 \cos(x)}{10 - x^2}$, show that the absolute value of the error when we approximate $f(1)$ using the second Maclaurin polynomial is less than $0.04 = 1/25$.

- 4 marks 12. (a) Let $g(x)$ be a continuous function for which $g'(x)$ and $g''(x)$ exist. If $g(x)$ has at least three zeros, then how many zeros must $g'(x)$ and $g''(x)$ have? Explain your answer carefully.

- 4 marks (b) Consider the equation $2x^2 - 3 + \sin(x) + \cos(x) = 0$. Show that this has at least two solutions.

- 2 marks (c) Show that the same equation cannot have more than two solutions. Part (a) will help you here.