

Be sure that this examination has 13 pages including this cover

The University of British Columbia  
Sessional Examinations - December 2012

Mathematics 100-180

*Differential Calculus with Applications to Physical Sciences and Engineering*

Closed book examination

Time: 2.5 hours

Last Name: \_\_\_\_\_ First Name: \_\_\_\_\_

Student Number: \_\_\_\_\_ Instructor's Name: \_\_\_\_\_

Signature: \_\_\_\_\_ Section Number: \_\_\_\_\_

**READ AND OBSERVE THE FOLLOWING EXAM RULES**

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
  - (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
  - (iii) purposely viewing the written papers of other examination candidates;
  - (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  - (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) — (electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

|       |  |     |
|-------|--|-----|
| 1     |  | 9   |
| 2     |  | 12  |
| 3     |  | 9   |
| 4     |  | 9   |
| 5     |  | 8   |
| 6     |  | 16  |
| 7     |  | 10  |
| 8     |  | 8   |
| 9     |  | 8   |
| 10    |  | 6   |
| 11    |  | 5   |
| Total |  | 100 |

Marks

**Short-Answer Questions.** Questions 1 – 4 are short-answer questions. Put your answers in the boxes provided. **Simplify your answers as much as possible, and show your work.** Each question is worth 3 marks, but not all questions are of equal difficulty.

- [9] 1. (a) Find a function  $f(x)$  that satisfies  $f'(x) = 2 \cos x - e^x$  and  $f(0) = 0$ .

|        |
|--------|
| Answer |
|--------|

- (b) Use a linear approximation to estimate  $(2.001)^4$ . Write your answer in the form  $n/1000$ , where  $n$  is an integer.

|        |
|--------|
| Answer |
|--------|

- (c) If  $y = (\sin x)^{\sin x}$ , find  $y'$ .

|        |
|--------|
| Answer |
|--------|

- [12] 2. (a) If  $f(x) = e^{(\sin x)^2}$ , find  $f'(x)$ .

Answer

- (b) Find the slope of the tangent line to the curve  $y + x \cos y = \cos x$  at the point  $(0, 1)$ .

Answer

- (c) If  $y = \sin^{-1}(\ln x)$ , find  $dy/dx$ . *Note:* Another notation for  $\sin^{-1}$  is  $\arcsin$ .

Answer

- (d) Let  $f(x) = g(2 \sin x)$ , where  $g'(\sqrt{2}) = \sqrt{2}$ . Find  $f'(\pi/4)$ .

Answer

- [9] **3.** (a) Suppose the tangent line to the curve  $y = f(x)$  at  $x = 1$  passes through the points  $(-2, 3)$  and  $(0, 5)$ . Find  $f(1)$  and  $f'(1)$ .

Answer

- (b) If  $4x - 9 \leq f(x) \leq x^2 - 4x + 7$  for  $x \geq 0$ , find  $\lim_{x \rightarrow 4} f(x)$ .

Answer

- (c) If  $f(1) = 3$ ,  $f$  is continuous on the interval  $[1, 4]$ , and  $f'(x) \leq -2$  for  $1 < x < 4$ , how large can  $f(4)$  possibly be?

Answer

- [9] 4. (a) Let  $a$  be a constant, and define

$$f(x) = \begin{cases} x^2 + a & \text{if } x \leq 1 \\ 2x - 3 & \text{if } x > 1 \end{cases}$$

Find the value of  $a$  for which  $f$  is continuous everywhere.

Answer

- (b) Evaluate  $\lim_{x \rightarrow 0} \frac{e^{3x^2} - 1}{\sin(x^2)}$

Answer

- (c) Evaluate  $\lim_{x \rightarrow \infty} \left(1 + \frac{5}{x}\right)^x$

Answer

**Full-Solution Problems.** In questions 5–11, justify your answers and **show all your work**. If a box is provided, write your final answer there. **Unless otherwise indicated, simplification of numerical answers is required in these questions.**

- [8] 5. A sample of the radioactive substance Rhodium-101 decayed to 12.5% of its original amount after 10 years.

(a) What percentage of the original amount is remaining after another 5 years?

|        |
|--------|
| Answer |
|--------|

(b) What is the half-life of Rhodium-101? Remember to simplify your answer completely.

|        |
|--------|
| Answer |
|--------|

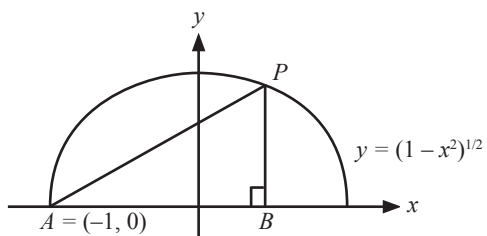
- [16] **6.** Let  $f(x) = 2x^{3/5} + 3x^{-2/5}$ . Note that the domain of  $f$  is the set of all nonzero real numbers; for example,  $f(-1) = -2 + 3 = 1$ .
- (a) Determine the interval(s) where  $f(x)$  is increasing, and the interval(s) where  $f(x)$  is decreasing.
- (b) Find the  $x$ -coordinates of the local maxima and local minima of  $f(x)$ .
- (c) Determine the interval(s) where  $f(x)$  is concave up and concave down, respectively. You may use:  $f''(x) = -(12/25)x^{-7/5} + (42/25)x^{-12/5} = (1/25)x^{-12/5}(-12x + 42)$ .
- (d) Find the  $x$ -coordinates of all inflection points of  $f(x)$ .

parts (e) and (f) on next page...

- (e) Find all vertical asymptotes of the graph  $y = f(x)$ .
- (f) Sketch the graph of  $y = f(x)$ , exhibiting all features in parts (a) – (e) and showing the  $y$ -coordinates of any points you found in part (b) (you do not need to include the  $y$ -coordinates of inflection points). Also, indicate the coordinates of all  $x$ -intercepts.



- [10] 7. Find the coordinates of the point  $P$ , which lies on the curve  $y = \sqrt{1-x^2}$ , in the diagram below for which the area of the right triangle  $ABP$  is the maximum. Remember to justify that your answer actually gives the *maximum* area.



|        |
|--------|
| Answer |
|--------|

Name:

December 2012

Mathematics 100-180

Page 10 of 13 pages

- [8] 8. A light sits atop a post  $H$  meters high, and a person of height  $h$ , where  $H > h$ , walks along a straight line away from the lamppost at a speed of  $v$  m/s. At what rate is the person's shadow lengthening?

Answer

- [8] **9.** Using the definition of the derivative, compute  $f'(x)$  if  $f(x) = \sqrt{1 - 2x^2}$ . Also, specify the domain of  $f'(x)$ . *No marks* will be given for the use of differentiation rules, but you may use them to check your answer.

Name:

December 2012

Mathematics 100-180

Page 12 of 13 pages

- [6] **10.** Find a function  $f(x)$  such that  $f'(x) = x^3$  and such that the line  $x + y = 0$  is tangent to the graph of  $y = f(x)$ .

Answer

- [5] **11.** Give a *complete* proof that for all  $x$  satisfying  $-1 \leq x \leq 1$ ,

$$0 \leq \cos(x) - \left(1 - \frac{x^2}{2}\right) \leq \frac{1}{24}$$

*Hint:* A question on Taylor polynomials has not yet appeared on this exam.