CUBIC HYPERSURFACES AND A CONJECTURE OF CASSELS AND SWINNERTON-DYER

Z. REICHSTEIN

Let $f(x_1, \ldots, x_n)$ be an irreducible homogeneous polynomial of degree d in n variables, with coefficients in a field K, where $\operatorname{char}(K)$ does not divide d. We would like to know if the equation $f = 0$ has a non-trivial solution in K^n or equivalently, if the hypersurface X cut out by f in $\mathbb{P}^{n-1}$ has a K-point. It is not difficult to show that f always has a solution in L^n for some finite field extension L/K of degree dividing d. Suppose we also know that f has a non-trivial solution in M^n for some finite field extension M/K of degree prime to d. Can we then conclude that f has a non-trivial solution in K^n?

For $d = 2$, the answer is "yes". This fact, known as Springer's Theorem, is of fundamental importance in the theory of quadratic forms and related areas. For $d \geq 4$, the answer is "no". There are even counterexamples in the case where even when X is a curve (i.e., $n = 3$).

The remaining case, $d = 3$, is a mystery. Cassels and Swinnerton-Dyer conjectured that the answer here is "yes", i.e., an analogue of Springer's theorem holds in this setting. This question is of interest to both number theorists, who are particularly concerned with the case where K is a number field, and algebraic geometers, who have observed that low degree complex hypersurfaces in projective space often exhibit interesting behavior and have studied families of such hypersurfaces in other contexts. There are also interesting connections to classical problems in Galois theory.

The conjecture of Cassels and Swinnerton-Dyer is known to be be true if X is a curve. There are also partial results in the case where D is a surface, due to Coray. An M-point on X may be viewed as a zero cycle of degree $[M : K]$, defined over K. Corey's idea was to decrease the degree of this cycle by placing it on a curve $C \subset X$, and then moving it using the Riemann-Roch theorem for curves. This leads to the followng partial solution to the conjecture of Cassels and Swinnerton-Dyer for surfaces: Suppose a cubic equation $f(x_1, x_2, x_3, x_4) = 0$ has a solution in M^n with $[M : K]$ prime to 3 then it has a solution in M_0^n for some field extension M_0/K of degree 1, 4 or 10.

The purpose of the project will be to learn (or review) the underlying mathematics, undertand Coray's proof and prove a variant of his result in higher dimensions ($n \geq 5$).