Summer 2016 NSERC USRA Report Forbidden Berge Hypergraphs

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This summer I worked with Anstee on forbidden Berge hypergraphs. Forbidden Berge hypergraphs are similar to forbidden configurations and patterns. We explored this problem using matrices with 0,1 entries. Define a matrix A to be simple if it is a $(0,1)$ matrix with no repeated columns. We can interpret the rows as vertices of a hypergraph and a column j as a hyperedge with a 1 in row i if vertex i is included in hyperedge j. Let F be a $(0,1)$ $k \times l$ matrix. We say that F is a *Berge hypergraph* of A and write $F \lll A$ if there is a $k \times l$ submatrix of A that has a row and column permutation, call it G, such that $G \geq F$. Let

$$\text{BAvoid}(m, \mathcal{F}) = \{\mathcal{A} : \mathcal{A} \text{ is } m\text{-rowed, simple, } \mathcal{F} \not\lll \mathcal{A} \text{ for all } \mathcal{F} \in \mathcal{F}\},$$

$$\text{Bh}(m, \mathcal{F}) = \max_{A}\{\|A\| : A \in \text{BAvoid}(m, \mathcal{F})\}.$$

$\text{Bh}(m, \mathcal{F})$ is the extremal function we are interested in. In forbidden configurations we write $F \prec A$ and say F is a configuration of A if there exists a row and column permutation a submatrix of A, say G with $F = G$. With forbidden configurations we have

$$\text{Avoid}(m, \mathcal{F}) = \{\mathcal{A} : \mathcal{A} \text{ is } m\text{-rowed, simple, } \mathcal{F} \not\prec \mathcal{A} \text{ for all } \mathcal{F} \in \mathcal{F}\},$$

$$\text{forb}(m, \mathcal{F}) = \max_{A}\{\|A\| : A \in \text{Avoid}(m, \mathcal{F})\}.$$

In most cases we consider $\mathcal{F} = \{F\}$ in which case we write $\text{Bh}(m, \mathcal{F}) = \text{Bh}(m, F)$. We establish a complete classification of all $\text{Bh}(m, F)$ where F is a $k \times l$ matrix with $k \leq 5$, with the exception of one missing maximal result for $k = 5$. We also provide

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many general results and draw an interesting connection to a problem explored by Alon and Shikhelman[1].

Theorem 0.1 *Let $I_k = K_k^1$ denote the $k \times k$ identity matrix. Then*

$$\mathrm{Bh}(m, I_k) = 2^{k-1}.$$

The classification for the $\Theta(1)$ and $\Theta(m)$ boundary follows from this result and a result in forbidden configurations. It is worth noting that $\mathrm{forb}(m, I_k)$, the extremal function of interest in forbidden configurations, is significantly different.

Lemma 0.2 *Given $A \in \mathrm{BAvoid}(m, F)$, there exists a matrix $T(A) \in \mathrm{BAvoid}(m, F)$ with $\|A\| = \|T(A)\|$ and $T_i(T(A)) = T(A)$ for $i = 1, 2, \ldots, m$.*

Let $\mathcal{S}$ be the set system of A; this lemma implies that $\mathcal{S}$ is a downset, i.e if $S \in \mathcal{S}$ and $S' \subset S$ then $S' \in \mathcal{S}$. A result of this lemma is that $\mathrm{Bh}(m, K_2 \times K_t) = \Theta(\mathrm{ex}(m, K_3^2, K_2 \times K_t))$, where $\mathrm{ex}(m, K_3, K_2 \times K_t)$ is the number of triangles in a graph avoiding the complete bipartite graph on 2 and t vertices. This specific problem, as well as the general problem of $\mathrm{ex}(m, G, F)$, the maximum number of subgraphs G in a graph avoiding F as a subgraph, has been addressed by Alon and Shikhelman in [1]. From our work this summer we believe the problems of interest to be $\mathrm{Bh}(m, F)$ where $F = I_{a_1} \times I_{a_2} \times \cdots \times I_{a_p}$, a generalization of the problem posed by Zarankiewicz[2].

We also obtained a classification of growth rates $\mathrm{forb}(m, F)$ where F is the vertex-edge incidence matrix of a forest. Let

$$H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Theorem 0.3 *Assume $k \geq 5$ and let F be the $k \times l$ vertex-edge incidence matrix of a forest T.*

1. $\mathrm{forb}(m, F)$ *is $\Theta(m^{k-3})$ if and only if $F \prec H_1$*
2. $\mathrm{forb}(m, F)$ *is $\Theta(m^{k-2})$ if and only if $F \not\prec H_1$ and T has at most 2 vertices of degree ≥ 3 and those two vertices are connected by a path of at most 2 or not connected.*
3. *if F is not one of the two previous cases, then* $\mathrm{forb}(m, F)$ *is $\Theta(m^{k-1})$.*

References

[1] Alon N, Shikhelman C. Many T copies in H-free graphs. Journal of Combinatorial Theory, Series B. 2016.

[2] Zarankiewicz K, Problem P 101, Colloq Math. 2:301 1951