Matrix coefficients of linear connected reductive groups and constructibility

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Abstract

Assuming as familiar the definition of an admissible representation of a linear connected reductive group, and that K-finite matrix coefficients of such representations are eigenfunctions of the universal enveloping algebra, we examine constructibility of matrix coefficients. While not matrix coefficients are in general not constructible, the real and complex parts of their associated τ-spherical functions can be viewed as linear combinations of functions $z \mapsto z^s$ with constructible coefficients. To this end we examine the monodromy of the system of differential equations associated to a τ-spherical function. After putting coordinates on the group in question, we obtain the above result. We conclude with a detailed example for $\mathrm{SL}(2,\mathbb{R})$.

Contents

1 Introduction

1.1 Project summary

This summer I worked with Prof. Julia Gordon to determine to what extent matrix coefficients of certain representations of Lie groups were constructible in the sense of [2]. This would allow an asymptotic expansion for matrix coefficients classically arrived at after much work to be obtained automatically via existing model theory results. The project sits within the fields of representation theory and model theory. The project was a continuation of the reading course I took under the supervision of Prof. Gordon during the fall and spring semesters of the previous academic year. It is this course that I learned the prerequisites listed in subsection 1.2; a summary of the major relevant results in contained in the report I completed for that course.

As stated in the abstract, we determined that in general neither matrix coefficients nor τ-spherical functions were constructible. Other potential avenues of research related to orbital integrals or working in logical language with an added symbol for each function $z \mapsto z^s$ where proposed, and we are currently in the preliminary stages of examining orbital integrals.

1.2 Statement of prerequisites

For reasons of economy of space, we assume the following background: Generalities about linear connected reductive groups G; the Peter-Weyl theorem and its consequences for arbitrary compact Lie groups; the universal enveloping algebra as the (associative) algebra of all differential operators on G; admissible representations of G and K-finite vectors, where $K \subset G$ is the subgroup fixed by the Cartan involution; and C^∞ vectors. The main result we will start from is

Theorem 1. *K-finite matrix coefficients for admissible irreducible representations of G are eigenfunctions for the centre of the universal enveloping algebra $Z(\mathfrak{g}^{\mathbb{C}})$.*

1.3 Notation

We will fix now the following notation:

Definition 1. We fix here the notation used throughout the rest of this essay.

1. $G =$ Our group of real of complex matrices stable under conjugate transpose;
2. $\mathfrak{g} =$ the Lie algebra of G, $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ when $\mathfrak{g}$ is a Lie algebra of real matrices;
3. $\Theta\colon G \to G$, $\Theta\colon g \mapsto \overline{g^{-1}}^{\mathrm{T}}$ is our *Cartan involution*;
4. $\theta\colon \mathfrak{g} \to \mathfrak{g}$, $\theta\colon X \mapsto -\overline{X}^{\mathrm{T}}$ is therefore the Cartan involution on $\mathfrak{g}$.
5. $\Sigma =$ the (restricted) *root system* for $\mathfrak{g}^{\mathbb{C}}$, $\Sigma^+ =$ the positive roots;
6. $\Delta =$ a *base* for Σ, *i.e.,* a choice of positive simple roots;

7. $\mathscr{W}$ = the *Weyl group* for Σ;

8. $K = \left\{g \in G \,\middle|\, (g^{\mathrm{tr}})^{-1} = g\right\}$, the fixed points of the Cartan involution;

9. $\mathfrak{k} = \{X \in \mathfrak{sl}(2,\mathbb{R}) \mid \theta(X) = X\}$, the Lie algebra for K;

10. $\mathfrak{p} = \{X \in \mathfrak{sl}(2,\mathbb{R}) \mid \theta(X) = -X\}$;

11. $\mathfrak{a}$ = a maximal abelian subspace of $\mathfrak{p}$;

12. $\mathfrak{a}^+$, $\mathfrak{a}^-$ = the positive and negative Weyl chambers, identified via the Killing form;

13. $M = Z_K(\mathfrak{a})$, the centraliser in K of $\mathfrak{a}$;

14. $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$, the Lie algebra for M;

15. $\mathfrak{n} = \oplus_{\lambda\in\Sigma^+}\mathfrak{g}_\lambda$. For the usual choices this makes $\mathfrak{n}$ strictly upper triangular, hence nilpotent.

16. A = group with Lie algebra $\mathfrak{a}$. After the usual choices, $A = \left\{\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & a_{nn}\end{pmatrix} \in G \,\middle|\, a_{ii} \in \mathbb{R},\ a_{ii} > 0\right\}$;

17. $A^\pm = \exp \mathfrak{a}^\pm$;

18. N = group with Lie algebra $\mathfrak{n}$. After the usual choices, $N = \left\{\begin{pmatrix} 1 & & & \\ 0 & 1 & * & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 1\end{pmatrix} \in G\right\}$;

Definition 2. In the case where $G = \mathrm{SL}(2,\mathbb{R})$, this notation specializes to, after making the standard choice of base for $\mathfrak{g}$,

1. $G = \mathrm{SL}(2,\mathbb{R})$;

2. $\Sigma = \left\{\alpha, -\alpha \,\middle|\, \alpha\colon \begin{pmatrix} t & 0 \\ 0 & -t\end{pmatrix} \mapsto 2t\right\}$;

3. $\Sigma^+ = \Delta = \{\alpha\}$;

4. $K = \mathrm{SO}(2) = \left\{g \in \mathrm{SL}(2,\mathbb{R}) \,\middle|\, (g^{\mathrm{tr}})^{-1} = g\right\}$;

5. $M = \left\{\begin{pmatrix} -1 & 0 \\ 0 & -1\end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\right\}$;

6. $A = \left\{\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \,\middle|\, t > 0\right\}$;

7. $A^+ = \left\{\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \,\middle|\, t > 1\right\} = \exp \mathfrak{a}^+$;

8. $A^- = \left\{\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \,\middle|\, 1 > t > 0\right\} = \exp \mathfrak{a}^-$;

9. $N = \left\{\begin{pmatrix} 1 & * \\ 0 & 1\end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})\right\}$;

10. $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C})$;

11. In $\mathfrak{g}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$;

12. $\mathfrak{k} = \{X \in \mathfrak{sl}(2,\mathbb{R}) \,|\, \theta(X) = X\} = \left\{ \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \,\middle|\, b \in \mathbb{R} \right\} = \mathfrak{so}(2)$;

13. $\mathfrak{p} = \{X \in \mathfrak{sl}(2,\mathbb{R}) \,|\, \theta(X) = -X\} = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \,\middle|\, b \in \mathbb{R} \right\}$;

14. $\mathfrak{m} = \emptyset$;

15. $\mathfrak{a} =$ diagonal matricies in $\mathfrak{sl}(2,\mathbb{R})$;

16. $\mathfrak{a}^+ = (0, \infty)$ as identified by the Killing form;

17. $\mathfrak{a}^- = (-\infty, 0)$ as identified by the Killing form;

18. $\mathfrak{n} =$ strictly upper triangular matricies in $\mathfrak{sl}(2,\mathbb{R})$.

2 Matrix coeffcients

2.1 τ-spherical functions

Let τ_i be representations of K on finite-dimensional vector spaces U_i for $i \in \{1, 2\}$.

Definition 3. A function F in $C^\infty(G, \mathrm{Hom}_{\mathbb{C}}(U_2, U_1))$ is *τ-spherical* if for all $k_1, k_2 \in K$,

$$F(k_1 x k_2) = \tau_1(k_1) F(x) \tau_2(k_2).$$

One remarks immediately that such functions are determined by their values on A, as we have the decomposition $G = KAK$. By smoothness, we in fact have that F is determined by its values on A^+, as $G^{(0)} = KA^+K$ is an open dense subset of G, because we can view the Weyl group as a subquotient of K and the Weyl group transitively on Weyl chambers. We set

$$C^\infty_\tau(G^{(0)}) = \left\{ F \in C^\infty(G^{(0)}, \mathrm{Hom}_{\mathbb{C}}(U_2, U_1) \,\middle|\, F(k_1 x k_2) = \tau_1(k_1) F(x) \tau_2(k_2) \right\}.$$

A function in C^∞_τ is the restriction to $G^{(0)}$ if and only if it extends to G as a smooth function.

A general procedure for producing τ-spherical functions from an admissible representation of G will not be needed, but we include one here for completeness. Let $\hat{K}$ be the set of K-types of a representation π on V, then partition a subset of K via S_1 and S_2, so that $S_1 \sqcup S_2 \subset \hat{K}$. We know

$$\pi \restriction_K = \bigoplus_{\omega \in \hat{K}} n_\omega \omega,$$

and we can set

$$\tau_i := \bigoplus_{\omega \in S_i} n_\omega \omega$$

and $\tau := (\tau_1, \tau_2)$. Let U_i be the vector space for τ_i, and let $E_i \colon V \twoheadrightarrow U_i$ be the resulting projections. Then

$$F(x) = E_1 \pi(x) E_2 \colon U_2 \to U_1$$

so F is τ-spherical. In practice, we shall be interested, when we apply theorem 6 and section 2.6 to $\mathrm{SL}(2,\mathbb{R})$ in section 3 , in the case where S_1 and S_2 are each a single K-type.

2.1.1 τ-radial components

Our ultimate goal is to calculate the effect of elements of $U(\mathfrak{g}^{\mathbb{C}})$ on F for τ-spherical F. A computation shows that if X is in the root space $\mathfrak{g}_\alpha$ for some root α, then X is a difference of an element of $\mathfrak{k}$ and an element of $\mathrm{Ad}(a)\mathfrak{k}$ for $a \in A^+$. The coefficients of the elements depend in general on a. The calculation works equally well for θX, and hence it shows that

$$\theta\mathfrak{n} + \mathfrak{n} \subset \mathfrak{k} + \mathrm{Ad}(a)\mathfrak{k}. \tag{1}$$

We know already that

$$\mathfrak{g} = \theta\mathfrak{n} + \mathfrak{a} + \mathfrak{m} + \mathfrak{n} \tag{2}$$

corresponding to

$$\mathfrak{g} = \text{lower triangular} + \text{diagonal} + \text{upper triangular}.$$

Together (1) and (2) prove

Proposition 1. *For fixed $a \in A^+$, we have*

$$\mathfrak{g} = \mathrm{Ad}(a^{-1})\mathfrak{k} + \mathfrak{a} + \mathfrak{k},$$

whence by the Poincaré-Birkhoff-Witt theorem, monomials

$$(\mathrm{Ad}(a^{-1})X)HY$$

are a basis for $U(\mathfrak{g}^{\mathbb{C}})$. Here $H \in U(\mathfrak{a}^{\mathbb{C}})$ and $Y, X \in U(\mathfrak{k}^{\mathbb{C}})$.

We start out towards our goal by calculating $(\mathrm{Ad}(a^{-1})X)HYF(a)$, and then show that derivatives of F are as determined by A^+ as F itself is. We have if X is a first-order differential operator

$$\begin{aligned}(\mathrm{Ad}(a^{-1})X)HYF(a) &= \frac{d}{dt}HYF(a\exp(\mathrm{Ad}(a^{-1})tX))\big|_{t=0} \\ &= \frac{d}{dt}HYF(a\exp(a^{-1}tXa))\big|_{t=0} = \frac{d}{dt}HYF(\exp(tX)a)\big|_{t=0}.\end{aligned}$$

Now for general k-th order $X \in U(\mathfrak{k}^{\mathbb{C}})$, we can calculate

$$\begin{aligned}(\mathrm{Ad}(a^{-1})X)HYF(a) &= \frac{d^k}{dt^k}HYF(\exp(tX)a)\big|_{t=0} &(3)\\ &= \frac{d^k}{dt^k}\frac{d^l}{dt_1^l}\frac{d^n}{dt_2^n}F(\exp(tX)a\exp(t_1H)\exp(t_2Y)\big|_{t=t_1=t_2=0} &(4)\\ &= \tau_1(X)HF(a)\tau_2(Y) &(5)\end{aligned}$$

by definition of all the representations involved. We can now extend by linearity and obtain an differential operator $D_\tau(u)$ from an element $u \in U(\mathfrak{g}^{\mathbb{C}})$. Then

$$uF(a) = D_\tau(F\restriction_{A^+})(a).$$

Definition 4. The operator $D_\tau(u)$ is the τ-*radial component* of u. It is a differential operator on A^+ with variable coefficients.

Each term in $D_\tau(u)$, as shown in (3), differentiates by H and multiplies on the left and right by τ_1 and τ_2, respectively. In this way each term is in $\mathrm{End}_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{C}}(U_2, U_1))$ for each a. Therefore as a function on A^+, each term is a map $A^+ \to \mathrm{End}_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{C}}(U_2, U_1))$.

We have now accomplished the first step towards the ultimate goal for this section mentioned above. The next step is to calculate how a first-order operator X in $\mathfrak{g}$ acts on $F(k_1ak_2)$. We have

$$XF(k_1xk_2) = \frac{d}{dt}F(k_1xk_2\exp tX) = \frac{d}{dt}F(k_1x\exp(t\mathrm{Ad}(k_2)X)k_2) = \tau_1(k_1)F(x\exp(\mathrm{Ad}(k_2)tX))\tau_2(k_2). \tag{6}$$

The above therefore also holds for monomials, and so

$$uF(k_1ak_2) = \tau(k_1)(uF)(a)\tau_2(k_2)$$

provided that, in light of (6), u is fixed by $\mathrm{Ad}(k)$. Thus in particular we have

Proposition 2. *For $u \in Z(\mathfrak{g}^{\mathbb{C}})$ and F in $C^\infty_\tau(G^{(0)})$, we have*

$$uF(k_1ak_2) = \tau_1(k_1)(uF)(a)\tau_2(k_2).$$

Thus uF in is $C^\infty_\tau(G^{(0)})$ too.

We have therefore achieved our ultimate goal for this section, and obtained the best news we could, in light of the connection between $Z(\mathfrak{g}^{\mathbb{C}})$ and K-finite matrix coefficients. Namely, uF is τ-spherical and so determined by its restriction to A^+.

Although 2 shows this next remark to be inconsequential for $\mathrm{SL}(2,\mathbb{R})$, the fact that M centralizes A implies that, for $m \in M$,

$$\tau_1(m)F(a) = F(a)\tau_2(m),$$

so actually $F(a) \in \mathrm{Hom}_M(U_2, U_1) \subset \mathrm{Hom}_{\mathbb{C}}(U_2, U_1)$.

2.1.2 τ-spherical functions made from matrix coefficients

In order to connect section 2.2 with this section for application in 2.4, we will need to associate matrix coefficients to τ-spherical functions and vice-versa. We can think of a K-finite (in both places) matrix coefficient as a function from G to $(U_1 \otimes U_2^*)^*$ for K-types U_1 and U_2. This is because the map $G \to \mathbb{C}$ sending $g \mapsto (\pi(g)u_1, u_2)$ is equivalent to a map $G \to (U_1 \otimes U_2^*)^*$ sending

$$g \mapsto ((u, v) \mapsto (\pi(g)u, v)). \tag{7}$$

The standard fact that $H \otimes_R J^* \simeq \mathrm{Hom}_R(J, H)$ for general R-modules J and H, and finite-dimensionality of U_1 and U_2 then say that (7) is equivalent to a map $G \to \mathrm{Hom}_{\mathbb{C}}(U_2, U_1)$.

The same thing is said more carefully in [5], where it is also easier to see why one side of the equivalence should actually be τ-spherical. There is an identification

$$\mathrm{Hom}_{K\times K}(U_2 \otimes U_1, C^\infty(G)) \longleftrightarrow C^\infty_\tau(G)'$$

where $C^\infty_\tau(G) \simeq C^\infty_\tau(G)'$, where the right side has the target replaced with $U_1^* \otimes U_2^*$ by finite-dimensionality, via

$$\langle F_\phi(g), u_1 \otimes u_2\rangle = \phi(u_1 \otimes u_2)(g) \tag{8}$$

for $F_\phi \in C^\infty_\tau(G)'$ and $\phi \in \mathrm{Hom}_{K\times K}(U_2 \otimes U_1, C^\infty(G))$. Note that this construction, in whichever phrasing, works for representations on Banach spaces equally well as for Hilbert spaces.

2.2 The fundamental matrix, the monodromy representation, and asymptotics

The perspective in this section is more explicit about the topological factors at play in determining the eventual form of the fundamental matrix arrived at in theorem 6. Specifically, we will develop the language to understand the statement that a *monodromy-invariant map from a covering space is truly a map from the topological space itself.* This is the perspective that is developed in [1].

2.2.1 Deck transformations and monodromy

While [1] as well as [4] both eventually specialize to the complex unit disk, both also seem to find it instructive to begin more generally. Let therefore X be a connected complex manifold based at x_0, and $\tilde{X}$ be the universal covering space, with covering map $p\colon \tilde{X} \to X$. The fundamental group $\pi_1(X, x_0)$ of X acts on $\tilde{X}$ by permuting elements within a fibre according to the definition below.

Definition 5. Let X be a topological space with fundamental group $\pi_1(X, x_0)$ and $\tilde{X}$ be a covering space for X. Let $\gamma \in \pi_1(X, x_0)$, and let $\tilde{\gamma}$ be its unique lift (which exists) with starting point $\tilde{x}$ for any $\tilde{x}$ in the fibre over $x \in X$. Define the *deck-transformation* $T_\gamma : \tilde{X} \to \tilde{X}$ by saying $T_\gamma(\tilde{x})$ is the terminal point of the loop $\tilde{\gamma}$.

Now in the notation of [1], let $\mathcal{O}_X(W)$ and $\mathcal{O}_{\tilde{X}}(W)$ be shaves of germs of holomorphic functions $X, \tilde{X} \to W$ for some finite-dimensional vector space W over $\mathbb{C}$. The action by deck transformations induces a representation of $\pi_1(X, x_0)$ on the vector space of holomorphic functions $f : \tilde{X} \to W$, via

$$T_\gamma^*(f) := f \circ T_\gamma.$$

Definition 6. T_γ^* is the *monodromy transformation corresponding to* γ.

Definition 7. Global sections of $p^{-1}\mathcal{O}(W)$, *ie*, holomorphic functions $\tilde{X} \to W$ will be called *multivalued sections of* $\mathcal{O}_X(W)$. See subsubsection 2.2.4.

This last definition begins to reconcile this abstract approach with that of [4], and we can now proceed to introduce differential equations.

2.2.2 The fundamental matrix

Let X now be an open connected subset of $\mathbb{C}^n$, $E = \mathrm{End}_{\mathbb{C}}(W)$, and $F_1, F_2, \ldots, F_n : X \to E$ be holomorphic. Consider the differential equation system

$$\partial_i \Phi = F_i \Phi \; i \in \{1, \ldots, n\}$$

on X. The Φ here are functions $X \to W$ (c.f. definition 3). A local solution on $U \subset X$ determines a subsheaf $\mathscr{S}$ of $\mathcal{O}_X(W)$, and global sections of $p^{-1}\mathscr{S}$ are multiple-valued local solutions. Let $\mathscr{S}_{x_0}$ be the stalk of germs of solutions at x_0. Define a subspace

$$W_0 := \{\varphi(x_0) \,|\, \varphi \in \mathscr{S}_{x_0} \text{ is the germ of a local solution at } x_0\}$$

of W. Linearity of the differential equations ensures that W_0 is closed under addition.

The above language is to set up a lemma to prove the following important theorem.

Theorem 2. *The map $p^{-1}\mathscr{S}_{\tilde{X}}(W) \to W_0$ sending $\Phi \mapsto \Phi(\tilde{x_0})$ is a linear isomorphism. Hence, local solutions of our system extend to global multi-valued solutions.*

The first step in the (omitted) proof is the lemma below. It shows injectivity (via stalk-wise injectivity) for the theorem. The second step involves lifting local solutions to the covering space $\{(x, \varphi) \,|\, \varphi \in \mathscr{S}_x, \; x, \in X\}$ of X.

Lemma 1. *The map $\mathscr{S}_{x_0} \to W_0$ sending $\varphi \mapsto \varphi(x_0)$ is a linear isomorphism.*

Now for the payoff:

Theorem 3. *Let $E_0 = \mathrm{Hom}_{\mathbb{C}}(W_0, W)$. By theorem 2, there is a holomorphic map $S : \tilde{X} \to E_0$ such that*

1. *$S(\tilde{x_0}) : W_0 \hookrightarrow W$;*
2. *$\forall w \in W_0$, the map $\tilde{X} \to W$ given by $x \mapsto S(x)w$ is a multiple-valued solution of the system.*

The map S is referred to by [4] and [1] as the *analogue of the fundamental matrix*. We shall just call it the *fundamental matrix*.

The global W-valued holomorphic solutions of the system in $p^{-1}\mathscr{S}(\tilde{X})$ form a vector space invariant under monodromy transformations; such transformations just permute the multiple values, sending solutions to solutions. Therefore for all $\gamma \in \pi_1(X, x_0)$, or equivalently for all T_γ, $T_\gamma^* S(\tilde{x})w$ is a new multiple-valued solution for all $w \in W_0$. Now theorem 2 says that multiple-valued solutions are in bijection with elements of W_0, so there must be a unique (invertible) $M_\gamma \in \mathrm{End}_{\mathbb{C}}(W_0)$ such that

$$T_\gamma^* S(\tilde{x}) = S(\tilde{x}) \circ M_\gamma.$$

Proposition 3. *The assignment $\gamma \mapsto M_\gamma$ is a representation of $\pi_1(X, x_0)$ on W_0,*

Proof. Clearly the identity acts as the identity endomorphism, as $T_e^* = \mathrm{Id}$, so theorem 2 says $M_e = \mathrm{Id}$. We recall first that the fundamental group acts on the covering space, so $\gamma \mapsto T_\gamma$ is a homomorphism. Then if $\gamma_1, \gamma_2 \in \pi_1(X, x_0)$, we have

$$S(\tilde{x}) \circ M_{\gamma_1\gamma_2} = T^*_{\gamma_1\gamma_2} S(\tilde{x}) = S(T_{\gamma_2^{-1}\gamma_1^{-1}}(\tilde{x})) = S(T_{\gamma_2^{-1}} T_{\gamma_1^{-1}}(\tilde{x})) = S(T_{\gamma_2^{-1}}(\tilde{x})) \circ M_{\gamma_1} = S(\tilde{x}) \circ M_{\gamma_1} \circ M_{\gamma_2}.$$

□

Definition 8. The representation above is the *monodromy* of the system.

2.2.3 Differential equations on the unit disk

The key reason to specialize to systems on a punctured disk is that the fundamental group will be abelian, and hence operate by communiting endomorphisms on W_0. This will allows for linear algebra games with the matrix exponential to put the fundamental matrix into a form from which we can read off the structure solutions. In particular, assuming the coordinates developed in section 2.3, one sees the structure of τ-spherical functions built from whatever matrix coefficients one pleases.

We therefore set $X = D^n \subset \mathbb{C}^n$,

$$X^\times = (D^\times)^k \times D^{n-k}, \tag{9}$$

where $D^\times = D \setminus \{0\} \subset \mathbb{C}$ and $1 \le k \le n$. Put

$$Y = X \setminus X^\times = \{x \,|\, x_i = 0 \text{ for some } i \in \{1, \dots, k\}\}. \tag{10}$$

We want to study the system

$$\partial_i \Phi = F_i \Phi$$

for $F_i \colon X^\times \to E$ holomorphic. Let $\mathbb{H} = \{x \in \mathbb{C} \,|\, \Re(z) < 0\}$ be the strict left half-plane in $\mathbb{C}$. The map

$$p\colon (x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_n) \mapsto (e^{x_1}, e^{x_2}, \dots, e^{x_k}, x_{k+1}, x_{k+2}, \dots, x_n)$$

is a covering map, and lets us identity the universal covering space $\widetilde{X^\times}$ with $\mathbb{H}^k \times D^{n-k}$.

For future use, we point out that as a real subspace $X^\times$ is semi-analytic, as defined in [3].

The fundamental group $\pi_1(X^\times, x_0) \simeq \mathbb{Z}^k$, with generators viewable as the loops around each puncture. Call each generator (taken in the counter-clockwise direction) to be γ_i, and consider the deck transformations $T_i := T_{\gamma_i}$. Now γ_i is a single loop around a puncture, so

$$T_i\colon (x_1, x_2, \dots, x_i, x_{i+1}, \dots, x_n) \mapsto (x_1, \dots x_{i-1}, x_i + 2\pi, x_{i+1}, \dots, x_n).$$

We have mentioned already that $\gamma \mapsto T_\gamma$ is a homomorphism, but the above formula is further proof that the T_i commute. It follows that if $M_i := M_{\gamma_i}$ for the monodromy representation on W_0, then the M_i commute too. Linear algebra now says there is a commuting family $\{R_j\}$ such that $M_j = \exp(-2\pi i R_i)$. (This lemma is proved for general families of commuting matrices in [4]. The idea is to use simultaneous Jordan-Chévally decomposition and form matrix logarithms.) This all proves the following

Theorem 4. *The map $P\colon \widetilde{X^\times} \to \mathrm{Hom}_{\mathbb{C}}(W_0, W)$ given by*

$$P\colon (x_1, \dots, x_n) \mapsto S(x_1, \dots, x_n) \exp(-(x_1 R_1 + \cdots + x_k R_k))$$

is invariant under all monodromy transformations T_γ^*. Hence it is a map $X^\times \to E_0$.

Corollary 1. *The fundamental matrix S has the form*

$$S(z) = P(z) z^R,$$

where we write formally that

$$z^R = (x_1, \dots, x_n)^R := \exp(z_1 R_1 + \cdots + z_l R_l).$$

S is a function on $\widetilde{X^\times}$, or $\mathbb{H}^k \times D^{n-k}$, and we see that all the multi-valuedness of S comes from z^R, or ultimately, from monodromy.

Corollary 2. *If we write* $z^{\mathbf{s}}(\log z)\mathbf{q}$ *for* $\mathbf{s} = (s_1, \dots s_l)$ *and* $\mathbf{q} = (q_1, \dots q_l)$ *to be the function*

$$(x_1, \dots, x_n) \mapsto \exp(s_1 x_1 + \cdots + s_l x_l)(x_1^{q_1} \cdot \cdots \cdot x_l^{q_1}),$$

the solutions with a fundamental matrix decomposed as in corollary 1 are of the form

$$\Phi(z) = \sum_{\mathbf{s}\in\mathcal{F}} \sum_{0\leq|\mathbf{q}|\leq q_0} z^{\mathbf{s}}(\log z)^{\mathbf{q}} \Phi_{s,q}(z)$$

for W*-valued holomorphic functions* $\Phi_{s,q}$.

Proof. We give a sketch of the rearrangement. Given that the R_i commute, we have

$$z^R = z_1^{R_1} \cdot \cdots \cdot z_l^{R_l},$$

and all we need to show is that

$$z_1^{R_1} \cdot \cdots \cdot z_l^{R_l} = \sum_{i=1}^{r} z^{s_i} p_i(\log z)$$

where p_i is function into $\mathrm{End}_{\mathbb{C}}(W_0)$ with polynomials in $\log z$ as its matrix coefficients.

By the Jordan-Chevally decomposition write simultaneously $R_i = D_i + N_i$, and let I_j be the projections onto the independent subspaces where all D_i are scalar operators. Then we define the *in general complex* number s_i as the number such that

$$D_i I_j = s_i^{(j)} I_j,$$

and define $s_i = (s_i^{(1)}, \dots, s_i^{(l)})$. Then one can calculate that

$$z_1^{D_1} \cdot \cdots \cdot z_l^{D_l} = \sum_{i=1}^{r} z^{s_i} I_i.$$

For the nilpotent parts, by definition

$$z_j^{N_j} = \exp(N_j \log z_j) = \sum_{k=0}^{n-1} (\log z_j)^k N_j^k$$

and so the product of the $z_i^{N_i}$ is a matrix with polynomial entries in $\log z$.

For a comment on the connection between q_0 and the size of the largest Jordan block in the fundamental matrix see subsection 3. □

There are several equivalent definitions of *simple singularities* depending on the available or desired technology. It will be sufficient for now to understand simple singularities by saying that

$$\begin{cases} z_i \partial_i \Phi = F_i \Phi & 1 \leq i \leq k \\ \partial_i \Phi = F_i \Phi & k < i \leq n \end{cases} \tag{11}$$

has simple singularities *along* Y. The fundamental matrix will, upon comparing this system to the one at the beginning of this subsection, have the form in corollary 2.

2.2.4 Aside for my own benefit on some constructions on sheaves.

What [1] is really talking when they make the definition equivalent to definition 7 is the inverse image functor from the the category of sheaves on X to the category of sheaves on $\tilde{X}$. In general, we have the following definition.

Definition 9. Let $f\colon X \to Y$ be continuous, and let $\mathscr{F}$ be a sheaf on Y. The *inverse image sheaf* is the sheaf obtained from the presheaf sending

$$U \mapsto \overset{\lim}{\to}_{V \supseteq f(U)} \mathscr{F}(V)$$

for open sets U of X. The direct limit is taken over all open V containing $f(U)$, which is in general not open. (Of course, we want to understand this for a covering map, which will locally be a homomorphism and hence, open after restricting to some neighbourhood. The hard way has got to be more rewarding, though.) The inverse image sheaf is written $f^{-1}\mathscr{F}$ (in [1], $f^*\mathscr{F}$).

Example 1 (Cribbed from Wikipedia)**.** If f is inclusion of a point y, then $f^{-1}\mathscr{F}$ is just the stalk of $\mathscr{F}$ at y, *ie*, the set of germs, the germs being exactly colimits over all the open sets containing y.

For the above definition to make sense, we need to actually understand how to make sheaves reliably from presheaves.

Definition 10. Given a presheaf $\mathscr{F}$, we define a sheaf $\mathscr{F}^+$ by

$$\mathscr{F}^+(U) := \left\{ s\colon U \to \bigcup_{P \in U} \mathscr{F}_P \right\}$$

such that for all $P \in U$, $s(P) \in \mathscr{F}_P$, and for all $P \in U$, there is a small neighbourhood of P, $V \subset U$ and $t \in \mathscr{F}(V)$ such that for all $Q \in V$, the germ $t_Q = s(Q)$. Further, there is a morphism of sheaves θ such that for any other morphism $\mathscr{F} \to \mathscr{G}$ for a sheaf $\mathscr{G}$, the map factor uniquely through $\mathscr{F}^+$.

We can check this construction actually gives a sheaf with the stated properties. It is clear that we get a presheaf with restriction of sets as the restriction maps. Further, if s maps points to the zero germ on every set of an open cover of an open set U, then s is the zero map; germs are local. For an open cover $\{V_i\}$ of U, and $\{s_i\}$ such that $s_i \restriction_{V_i \cap V_j} = s_j \restriction_{V_j \cap V_i}$ for all i, j, then this just says $s_i(P) = s_j(P)$ are the same germs for $P \in V_i \cap V_j$. We can therefore define

$$s\colon P \mapsto s_i(P) \restriction_{\cap_{V_j \ni P} V_j}$$

and this is well-defined. That $s(P) \in \mathscr{F}_P$ for all P follows from the s_i doing this; and the second requirement follows from each s_i being locally given by a section of $\mathscr{F}$, and then restricting to the intersection $\cap_{V_i \ni P} V_i$. For the map, we can set

$$\theta\colon s \mapsto (P \mapsto s_P).$$

The rest of the universal property requires properties of maps of stalks.

All this is to allow us to comfortably say that the sheaf of germs $\mathcal{O}_{\tilde{X}}(W) = p^{-1}\mathcal{O}_X(W)$, as for $U \subset \tilde{X}$,

$$p^{-1}\mathcal{O}_X(W)(U) = \overset{\lim}{\to}_{V \supseteq p(U)} \mathcal{O}_X(W)(V) = \mathcal{O}_X(W)(p(U)) = \{s\colon p(U) \to W\}$$

because for small enough U, p is open. There's a clear way to relate germs of functions on the left to germs of functions on $\tilde{X}$, namely, composition with the local homomorphism p, so the sheaf associated to the presheaf $p^{-1}\mathcal{O}_X(W)$ is isomorphic to the sheaf associated to $\mathrm{O}_{\tilde{X}}(W)$ (as a presheaf), but then this is just isomorphic to $\mathrm{O}_{\tilde{X}}(W)$.

2.3 Coordinates on the group

We have now listed all the differential equations we shall require, at the level of detail we shall require, and introduced τ-spherical functions. To connect them, we need coordinates on G, or at least on some subset of A, putting is into the setting of subsection 2.2.3. [4] and [1] differ on which subset of A they chose, [4] choosing A^+, the exponentials of the positive Weyl chamber (identified via the Killing form) $\mathfrak{a}^+$ and [1] choosing A^-. The primary difference in notation between [4] and [1] is a matter of deciding at what point to take exponentials and where to put a minus sign:

2.3.1 The set-up in [4]:

Recall $\Delta = \{\alpha_1, \ldots, \alpha_l\} \subset \Sigma$ to be the simple restricted simple roots of $\mathfrak{g}$ (*i.e.*, that Δ is a base for Σ). In particular, each α_i is in $\mathfrak{a}^*$. Let $\mathfrak{a}^+$ be identified with the positive Weyl chamber, and let $A^+ = \exp \mathfrak{a}^+$. Define coordinates

$$\iota\colon A \to (\mathbb{R}^+)^l, \quad \iota(\exp H) = (e^{-\alpha_1(H)}, e^{-\alpha_2(H)}, \ldots, e^{-\alpha_n(H)}) \tag{12}$$

for $a = \exp H$.

Example 2. In terms of of $\mathrm{SL}(2,\mathbb{R})$, $a = \mathrm{diag}(t, t^{-1})$ and

$$\iota(a) = e^{-2\log t} = t^{-2}.$$

Example 3. For $\mathrm{SL}(3,\mathbb{R})$, $a = \mathrm{diag}(t_1, t_2, (t_1 t_2)^{-1})$ and for the root $\alpha\colon \mathrm{diag}(u_1, u_2, -u_1 - u_2) \mapsto u_1 - u_2$, we get

$$e^{-\alpha \log a} = e^{-(\log t_1 - \log t_2)} = t_2/t_1.$$

Then $\iota\colon A^+ \stackrel{\sim}{\to} (0,1)^l$, as by definition, α_i will take positive values on the positive Weyl chamber.

2.3.2 The set-up in [1]:

Fix a set Δ of "simple restricted roots," now meaning *multiplicative characters* $A \to \mathbb{R}^\times_{>0}$ and use them to embed $A \hookrightarrow \mathbb{C}^{\#\Delta} = \mathbb{C}^l$. Now recall the meaning of the number $1 \leq k \leq n$ defined in (9). [4] makes silently the choice $k = n$, but [1] keeps this generality, defining a larger set of characters thus:

Definition 11. Let Λ be the set of *multiplicative characters* $\bar{\alpha}\colon A \to \mathbb{R}^\times_{>0}$ such that

1. The base $\Delta \subset \Lambda$;
2. The characters in $\Lambda \setminus \Delta$ are trivial ($\equiv 1$) on $A \cap [G, G]$.

   Note that for $G = \mathrm{SL}(n,\mathbb{R})$ and $G = \mathrm{SL}(n,\mathbb{C})$, $G = [G, G]$. This requirement seems to say that Λ contains no exponentials of restricted roots of $\mathfrak{g}$ not in Δ.
3. The differentials of the characters λ for a basis for $\mathfrak{a}^*$.

Remark 1. Stipulation 2 above deals with the centre of groups more general than the LCR groups we consider. It is relevant for disconnected groups like $\mathrm{GL}(n,\mathbb{R})$. In this case it guarantees stipulation 3, as the Lie algebra is sensitive only to the connected component of the group containing the identity.

Example 4. [1] gives examples of "roots" for $G = \mathrm{SL}(3,\mathbb{R})$, putting for the *actual root* α *of* $\mathfrak{g}$ given by

$$\alpha\colon \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & (-t_1 - t_2) \end{pmatrix} \mapsto t_1 - t_2, \tag{13}$$

the *multiplicative character* in Λ

$$\bar{\alpha}\colon \begin{pmatrix} e^{t_1} & & \\ & e^{t_2} & \\ & & e^{-t_1 - t_2} \end{pmatrix} \mapsto e^{t_1}/e^{t_2}. \tag{14}$$

We can expand upon this to illustrate requirement 3 above. Viewing $\mathfrak{a}^*$ as the space of 1×2 rows, we have $\alpha = (1, -1)$. Then the differential of $\bar{\alpha}$ at the identity is

$$\left(\frac{\partial \bar{\alpha}}{\partial t_1}, \frac{\partial \bar{\alpha}}{\partial t_2}\right)\Big|_{t_1=t_2=0} = \left(e^{t_1}/e^{t_2}, -e^{t_1}/e^{t_2}\right)\Big|_{t_1=t_2=0} = (1, -1).$$

There is then an embedding $\bar{\iota}\colon A \to \mathbb{C}^{\#\Lambda}$ by $a \mapsto (\lambda_1(a), \lambda_2(a), \dots)$.

The key point is that if restrict ourselves to the *negative* Weyl chamber $\mathfrak{a}^-$ and its exponential A^-, and set $k = n$, then

$$\bar{\iota}(A^-) = (0,1)^l$$

too.

In neither case has the punctured disk $D^\times$ yet been brought into play, which we know to be roughly half of the loose characterization we gave in (11). In both sources a fair amount of technicality is involved in doing this, and we will summarize it quickly in the section below.

2.4 Constructibility of coefficients

2.4.1 τ-spherical functions and differential equations

We are now nearly read to invoke the results of corollary 2 and obtain asymptotics of τ-spherical functions.

Theorem 5. *The following statements about differential operators in coordinates on A hold:*

1. *Let H_i be a dual basis for $\mathfrak{a}$, so normalized such that $\alpha_i(H_j) = \delta_{ij}$. Then under (either of) the coordinate system(s) developed in the last section, H_i as a differential operator on A corresponds to $-z_j \frac{\partial}{\partial z_j}$, where z_j is the coordinate in $\mathbb{C}^l$ corresponding to (the exponentiated root or multiplicative character) α_j.*

2. *For $Z \in Z(\mathfrak{g}^{\mathbb{C}})$, the τ-radial component $D_\tau(Z)$ corresponds to a element in the algebra of differential operators on D^l*

$$\mathcal{D}^* = \left\{ \sum_{k \in (\mathbb{Z}^+)^l \text{ finite}} A_k (z\partial)^k \,\middle|\, A_k \colon D^l \to \operatorname{End}_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}}(U_2, U_1) \text{ holomorphic} \right\}.$$

3. *The operator $D_\tau(Z) - D_\tau(\mu_{\Sigma^+}(Z))$ corresponds to an operator $d \in \mathcal{D}^*$ with coefficients A_k vanishing at 0. Here*

$$\mu_{\Sigma^+} \colon Z(\mathfrak{g}^{\mathbb{C}}) \to U(\mathfrak{k}^{\mathbb{C}})$$

is the projection made possible by the decomposition

$$\mathfrak{g} = \theta\mathfrak{n} \oplus \mathfrak{k} \oplus \mathfrak{n}.$$

4. *The system on D^l, $\{D_\tau(Z)(F\restriction_{A^+}) = 0 \mid Z \in I\}$, for some ideal of finite codimension $I \subset Z(\mathfrak{g}^{\mathbb{C}})$ has a simple singularity along $D^l - (D^\times)^l$. (c.f. the prototypical system (11).)*

Proof. We sketch the proof. The proof of (1) is a calculation directly from the definitions. (2) follows somewhat intuitivly from (1) but the proof is more involved and uses the twisted version μ_{Σ^+} of the Harish-Chandra homomorphism. For (3) one needs to show that the ideal $\mathcal{I} = \langle D_\tau(Z),\ Z \in I\rangle$ has finite codimension. One essentially uses that for $U(\mathfrak{h}^{\mathbb{C}})$, the isomorphic image of the Harish-Chandra homomorphism, the Weyl-fixed elements $U(\mathfrak{h}^{\mathbb{C}})^{\mathscr{W}}$, are a $U(\mathfrak{h}^{\mathbb{C}})$-module of finite rank. □

While this theorem is amalgamated from the development as in [4], [1] proceeds practically identically. In fact, their paper served as a basis for [4].

We can now invoke all the results of section 2.2, namely corollary 2:

Theorem 6. *Let π be an admissible representation of G on V, and let v, v' be K-finite vectors. We can assume each lies in a single K-type τ_1 or τ_2 operating on subspaces U_1 and U_2 of V, and form projections $E_i\colon V \to U_i$. Then*

$$F(x) = E_1\pi(x)E_2\colon G \to \mathrm{Hom}_{\mathbb{C}}(U_2, U_1)$$

is $\tau = (\tau_1, \tau_2)$-spherical, and by theorem 5 and corollary 2,

$$(F \restriction_{A^+} \circ \iota^{-1})(z) = \sum_{\mathbf{s}\in\mathcal{F}} \sum_{0\leq|\mathbf{q}|\leq q_0} z^{\mathbf{s}}(\log z)^{\mathbf{q}} F_{\mathbf{s},\mathbf{q}}(z).$$

Here $\mathcal{F} \subset \mathbb{C}^l$ is finite, $F_{\mathbf{s},\mathbf{q}}\colon D^l \to \mathrm{Hom}_{\mathbb{C}}(U_2, U_1)$ are holomorphic on all of D^l, and $q_0 \in \mathbb{Z}$.

Reading the multi-index notation, then, we see that the real and complex parts of an entry of the matrix $(F \circ \iota^{-1})(z)$ are each in the $\mathbb{R}$-algebra generated by real-analytic functions on D^l, the real natural logarithm $\log\colon (0, \infty) \to \mathbb{R}$, and the real part of w^u for complex number w and u.

2.5 Constructible functions

Here we summarize the definitions we require from the frameworks of [3] and then [2]. The definitions are developed for in general for the category of analytic real manifolds, but we shall need only the trivial cases $\mathbb{C}^n$ and G.

Definition 12. A subset S of a (Haussdorf, analytic) manifold M is *semi-analytic at $x \in M$* if there is an open neighbourhood U of x such that

$$U \cap S = \bigcup_{\text{finite}} \{y \in M \mid f(y) = 0, g_1(y) > 0, \ldots, g_n(y) > 0\}$$

where $f, g_1, \ldots, g_n$ are all analytic functions on U. S is *semi-analytic* if it is semi-analytic everywhere on M.

Definition 13. $S \subset M$ is *sub-analytic at $x \in M$* if S is locally a projection of a pre-compact semi-analytic set. Precisely, if there exists a neighbourhood U of x in M such that

$$U \cap S = U \cap \pi(S'),$$

where $S' \subset M \times \mathbb{R}^N$ is pre-compact and semi-analytic. S is *sub-anlalytic* if it is sub-analytic everywhere on M.

In particular, one sees that semi-analytic sets are sub-analytic.

Definition 14. A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is *sub-analytic* if it is definable by analytic functions restricted to compact cubes within their domains of analyticity, polynomial on all of $\mathbb{R}^n$, and quotients and compositions of them. The definitions may also include $=$, $<$, and Boolean operators.

Definition 15. The above definitions are all from [3]. This last and most directly applicable one is from [2]. A function on a sub-analytic domain $U \subset \mathbb{R}^n$ is *constructible* if it is an element of the $\mathbb{R}$-algebra generated by sub-analytic functions $U \to \mathbb{R}$ and functions $\log f(x)$, where $f\colon U \to (0, \infty)$ is sub-analytic.

The primary results we sought to apply are as follows. The first requires some particular coordinates to introduced in order to have a precise statement, so we give a rougher summary.

Theorem 7 (Theorem 2.9, [2], Preparation of constructible functions)**.** *If $f\colon X \times \mathbb{R}^n \to \mathbb{R}$ is constructible and X is a sub-analytic set, with $f(x, \cdot)$ Lebesgue-integrable on $\mathbb{R}^n$ for all $x \in X$, then there exists a partition of $\mathbb{R}^n$ and certain coordinates such that on each cell of the partition, f is a finite sum of functions of the form*

$$g(x)|\tilde{y}|^{\alpha}(\log|\tilde{y}|)^{\beta} u_j(x, y)$$

where $\tilde{y}$ is y in the coordinates mentioned above, α is a multi-index of rational numbers, β is a multi-index of natural numbers, and u_j is rational. Further, the expressions above have constant sign on their respective cells of the partition.

Theorem 8 (Theorem 2.5, [2], Stability under integration). *If $f\colon X \times \mathbb{R}^n \to \mathbb{R}$ is constructible and X is a sub-analytic set, with $f(x,\cdot)$ Lebesgue-integrable on $\mathbb{R}^n$ for all $x \in X$, then $F\colon X \to \mathbb{R}$ with*

$$F(x) = \int_{R^n} f(x,y)dy$$

is constructible.

It is this theorem that we hope to employ to show certain orbital integrals are constructible, and hoped to use to show that matrix coefficients were constructible, as many representations are realizable with integration over $\mathbb{R}^n$ as the inner product. For an example of such a representation, and the obstruction we encountered, see section 3.

2.6 Extent of Constructibility

The functions in theorem 6 are *not* constructible in the sense of [2]. Specifically, neither the real and complex parts of complex exponentiation are constructible. Indeed, one has

$$\Re w^{y+xi} = \Re e^{y \log z} e^{ix \log z},$$

and this last term has real part $\cos(x \log z)$. We cannot view $\log z$ as a restriction of log to a closed interval not including 0 without losing asymptotic significance, and so we cannot then compose cos with log. Additionally, we could not define cos as any restriction to a compact interval of the usual analytic function $\cos\colon \mathbb{R} \to \mathbb{R}$. In supposed analogy with the case for p-adic groups, the next best thing is true, though.

Theorem 9. *For F as in 6 and let $1 > \epsilon > 0$. Then on $\left\{z \in \mathbb{C}^l \,\middle|\, |z| \le 1-\epsilon\right\}$,*

$$(F \restriction_{A_\epsilon^+} \circ \iota^{-1})(z)$$

is a linear combination of functions $z^{\mathbf{s}}$ with constructible coefficients $\left\{z \in \mathbb{C}^l \,\middle|\, |z| \le 1-\epsilon\right\} \to \mathbb{R}$, after taking real of complex parts. Here

$$A_\epsilon^+ = \left\{\exp H \in \mathfrak{a} \,\middle|\, \alpha_i(a) \ge \log\frac{1}{\epsilon}\ \forall i \in \{1,\ldots,l\}\right\} \subset A.$$

Proof. The set A_ϵ^+ is in fact semi-analytic for any $1 > \epsilon > 0$, as

$$A_\epsilon^+ = \left\{a \in A \,\middle|\, \alpha_i(\log a) \ge \log\frac{1}{\epsilon}\ \forall i \in \{1,\ldots,l\}\right\} \tag{15}$$

$$= \left\{a \in A \,\middle|\, \alpha_i(\log a) > \log\frac{1}{\epsilon}\ \forall i \in \{1,\ldots,l\}\right\} \cup \bigcup_{j=1}^{l} \left\{a \in A \,\middle|\, \alpha_i(\log a) = \log\frac{1}{\epsilon}\ \forall i \le j,\ \alpha_i(\log a) > \log\frac{1}{\epsilon}\ \forall i > j\right\}. \tag{16}$$

Each set in the union is semi-analytic, as α_i is linear and log is analytic, so the union is semi-analytic. Now set

$$C_{\mathbf{s},\mathbf{q}}(z) = (\log z)^{\mathbf{q}} \underline{F}_{\mathbf{s},\mathbf{q}}(z).$$

This function is constructible, as $\underline{F}_{\mathbf{s},\mathbf{q}}$, defined by $F_{\mathbf{s},\mathbf{q}}$ restricted to the l-fold product of closed radius $(1-\epsilon)$-disks is restricted analytic, and log is constructible by definition. Then

$$(F \restriction_{A_\epsilon^+} \circ \iota^{-1})(z) = \sum_{\mathbf{s} \in \mathcal{F}} \sum_{0 \le |\mathbf{q}| \le q_0} z^{\mathbf{s}} C_{\mathbf{s},\mathbf{q}}(z)$$

has constructible coefficients. □

We would also now like to obtain a formula for converting theorem 6 into a theorem about matrix coefficients. Defining $F(x)$ as in the statement of that theorem, we note that the matrix coefficient

$$(\pi(g)v', v) = (F(x)v', v).$$

We can apply theorem 6 term-by-term and obtain for $x \in A^+$,

$$(F(x)v', v) = \left(\sum_{\mathbf{s}\in\mathcal{F}} \sum_{0\le|\mathbf{q}|\le q_0} \iota(x)^{\mathbf{s}} (\log \iota(x))^{\mathbf{q}} F_{\mathbf{s},\mathbf{q}}(\iota(x))v', v \right) = \sum_{\mathbf{s}\in\mathcal{F}} \sum_{0\le|\mathbf{q}|\le q_0} \iota(x)^{\mathbf{s}} (\log \iota(x))^{\mathbf{q}} \left(F_{\mathbf{s},\mathbf{q}}(\iota(x))v', v\right). \tag{17}$$

Remark 2. There are examples where ι is a constructible function, at least on A^-. For example, when $G = \mathrm{SL}(2,\mathbb{R})$, $\iota\colon \mathrm{diag}(t, t^{-1}) \mapsto t^2$. Whether the inner product should be constructible is more delicate. In fact, the next extended example will show there are situations even for $\mathrm{SL}(2,\mathbb{R})$ where v' and v need not be constructible functions, so any constructibility of integrals involving them would be far from an automatic consequence of theorem 8.

2.6.1 Example of a principal series representation with non-constructible K-finite vectors

The first thing we do is realise the unitary irreducible representation $\mathscr{P}^{\pm,iv}$, whose vector space we view as "truly" being $L^2(\mathbb{R})$, in another way.

Proposition 4 ([4],Exercise 10, §8, II.)**.** *Let $v \in \mathbb{R}$. Let $P^{+,iv}$ be the representation of $\mathrm{SL}(2,\mathbb{R})$ with Hilbert space*

$$H_v^+ = \left\{ f\colon \mathbb{R}^2 \to \mathbb{C} \,\middle|\, F(tx, ty) = |t|^{-1-iv} F(x,y),\ \|F\|_v < \infty \right\},$$

norm

$$\|F\|_v^2 = \frac{1}{2\pi} \int_0^{2\pi} |F(\cos\theta, \sin\theta)|^2 \, d\theta,$$

and action

$$P^{+,iv}(g)F(x,y) = P^{+,iv}(g)F\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = F\left(g^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right).$$

Then $P^{+,iv}$ is unitarily (up to a constant) equivalent with $\mathscr{P}^{+,iv}$.

Proof. Define $L\colon H_v^+ \to L^2(\mathbb{R})$ by $F(x,y) \mapsto F(1,y) := f(y)$. It is obvious that L is linear, but there are several things to check.

1. *L is actually into $L^2(\mathbb{R})$*: Let $F \in H_v^+$ and then we can estimate $\|LF\|_2^2$ by

$$\begin{aligned} \|LF\|_2^2 &= \int_{\mathbb{R}} |F(1,x)|^2 \, dx \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |F(1,\tan\theta)|^2 \frac{1}{\cos^2\theta} \, d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \left| \frac{1}{\cos\theta} \right|^{-1-iv} \right|^2 F(1,\tan\theta) \, d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(\cos\theta, \sin\theta) \, d\theta \qquad (18) \\ &\le \|F\|_v^2 < \infty. \end{aligned}$$

2. *L is injective:* If LF is the zero function, then $F(1,y) = 0$ for all y, so for all $x \neq 0$ we have $F(x,y) = |x|^{-1-iv} F(1, y/x) = 0$ and F is the zero function at least everywhere except the y-axis. This is a set of measure zero, so F is the zero function.

3. *L is an intertwining operator:* Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so that $g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. We calculate that

$$L(P^{+,iv}(g)F)(y) = L\left(F\left(g^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right)\right) = L\left(F\left(\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right)\right)$$
$$= L\left(F\left(dx - by, -cs + ay\right)\right) = F(d - by, -c + ay). \quad (19)$$

On the other hand,

$$\mathscr{P}^{+,iv}(g)(LF)(y) = \mathscr{P}^{+,iv}(g)f(y) = |-by+d|^{-1-iv} f\left(\frac{ay-c}{-by+d}\right)$$
$$= |-by+d|^{-1-iv} F\left(1, \frac{ay-c}{-by+d}\right) = F(-by+d, ay-c). \quad (20)$$

4. *L is surjective:* We note that H_v^+ is non-empty because it contains $F \equiv 1$, as

$$\|F\|_v^2 = \frac{1}{2\pi}\int_0^{2\pi} 1\, d\theta = 1 < \infty.$$

Surjectivity now follows from the fact that L is an injective intertwining operator into an irreducible representation.

5. *L is bounded:* This will follow from L being unitary up to a constant factor, hence an isometry up to a constant factor.

6. *L is unitary up to a constant factor:* From the definition,

$$\begin{aligned}
(LF, LG)_2 &= \int_{\mathbb{R}} F(1,x)\overline{G(1,x)}\, dx \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(1,\tan\theta)\overline{G(1,\tan\theta)}\frac{1}{\cos^2\theta}\, d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left|\frac{1}{\cos\theta}\right| F(1,\tan\theta) \left|\frac{1}{\cos\theta}\right| \overline{G(1,\tan\theta)}\, d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\cos\theta|^{-1-iv}||\cos\theta|^{iv}F(1,\tan\theta)|\cos\theta|^{-1+iv}|cos\theta|^{-iv}\overline{G(1,\tan\theta)}\, d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(\cos\theta,\sin\theta)\overline{G(\cos\theta,\sin\theta)}|\cos\theta|^{-iv}|\cos\theta|^{iv}\, d\theta \\
&= \frac{1}{2}\int_0^{2\pi} F(\cos\theta,\sin\theta)\overline{G(\cos\theta,\sin\theta)}\, d\theta \\
&= \pi(F,G)_v. \qquad (21)
\end{aligned}$$

□

We now attempt to define the correct eigenfunctions of K for $P^{+,iv}$. Let

$$\varepsilon_v^n(x,y) = \varepsilon_v^n(t\cos\vartheta, t\sin\vartheta) := e^{in\vartheta}|t|^{-1-iv}. \quad (22)$$

Proposition 5. *Let ε_v^n be as above. Then*

(a) ε_v^n is in H_v^+. Moreover, $\|\varepsilon_v^n\|_v = 1$;

(b) If $m \neq n$, then $(\varepsilon_v^m, \varepsilon_v^n)_v = 0$;

(c) ε_v^n is an eigenfunction of K.

Proof. (a) follows directly from the calculation

$$\|\varepsilon_v^n\|_v^2 = \frac{1}{1\pi}\int_0^{2\pi} |\varepsilon_v^n(\cos\vartheta, \sin\vartheta)|^2 \, d\vartheta = \frac{1}{\pi}\int_0^{2\pi} \left|e^{in\vartheta}\right|^2 \, d\vartheta = \frac{1}{2\pi}\int_0^{2\pi} d\vartheta = 1.$$

Likewise we have (b), as

$$(\varepsilon_v^n, \varepsilon_v^m)_v = \frac{1}{2\pi}\int_0^{2\pi} \varepsilon_v^n(\cos\vartheta, \sin\vartheta)\overline{\varepsilon_v^m(\cos\vartheta, \sin\vartheta)} \, d\vartheta = \frac{1}{\pi}\int_0^{2\pi} e^{i(n-m)\vartheta} \, d\vartheta = 0$$

if $m \neq n$. Finally, if k in K is rotation by ξ, then

$$\begin{aligned} P^{+,iv}(k)\varepsilon_v^n(x,y) &= P^{+,iv}(k)\varepsilon_v^n(t\cos\vartheta, t\sin\vartheta) \\ &= \varepsilon_v^n\left(k^{-1}\begin{pmatrix} t\cos\vartheta \\ t\sin\vartheta \end{pmatrix}\right) \\ &= \varepsilon_v^n(t\cos(\vartheta-\xi), t\sin(\vartheta-\xi)) \\ &= e^{in(\vartheta-\xi)}|t|^{-1-iv} \\ &= e^{-in\xi}\varepsilon_v^n(x,y). \end{aligned}$$

□

These specific calculations are actually viewable as a corollary of the following equally trivial observation.

Proposition 6. *Each ε_v^n, restricted to the circle in $\mathbb{R}^2$, is a character of the circle group, viewed as lying in $\mathbb{R}^2$. They are therefore eigenfunctions of the left regular representation of the circle.*

Proof.

$$\varepsilon_v^n(\cos(\vartheta+\xi), \sin(\vartheta+\xi)) = e^{in(\vartheta+\xi)} = e^{in\vartheta}e^{in\xi} = \varepsilon_v^n(\cos\vartheta, \sin\vartheta) \cdot \varepsilon_v^n(\cos\xi, \sin\xi).$$

□

Proposition 5 now follows from elementary Fourier analysis.

Corollary 3. *The functions in* (22) *are the* only *eigenfunctions of K.*

Proof. Functions in H_v^+ are determined by their restrictions to the circle, and these restrictions need to be characters of the circle group. □

Lemma 2. *Restricted to the unit circle in $\mathbb{R}^2$, the real and imaginary parts of each ε_v^n are constructible functions. In fact, each is a sub-analytic function.*

Proof. Trivial, as

$$\varepsilon_v^n(\cos\vartheta, \sin\vartheta) = e^{in\vartheta} = \cos(n\vartheta) + i\sin(n\vartheta)$$

is holomorphic, so its real and imaginary parts are each real-analytic, and viewable as restricted to $[0, 2\pi/n]$. □

Remark 3. Warning! The lemma above holds only for H_v^+ as a vector space! As a representation, we cannot view angles as restricted to $[0, 2\pi/n]$, as K must be able to act by rotations. We cannot fix this by precomposing with a quotient map either, as such a function has a saw-tooth-shaped graph made of countably, but *not finitely* many linear segments and is therefore not subanalytic.

We now calculate the value of $\varepsilon_v^n(t\cos\vartheta, t\sin\vartheta)$ for t other than 1. This will allow us to analyse $(P^{+,iv}(a)\varepsilon_v^n\restriction_{S^1})(x,y)$ away from the unit circle. In particular, we will know what acting by $a \in A^+$ does, as A^+ does not send the circle to itself. We have

$$\begin{aligned}
\varepsilon_v^n(t\cos\vartheta, t\sin\vartheta) &= |t|^{-1-iv}e^{in\vartheta} && (23)\\
&= |t|^{-1}|t|^{-iv}(\cos n\vartheta + i\sin\vartheta) && (24)\\
&= t^{-1}(\cos(-v\log t) + i\sin(-v\log t))(\cos(n\vartheta) + i\sin(n\vartheta)) && (25)\\
&= t^{-1}(\cos(v\log t)\cos(n\vartheta) + i\sin(n\vartheta)\cos(-v\log t) && (26)\\
&\qquad + i\sin(-v\log t)\cos(n\vartheta) - \sin(-v\log t)\sin(n\vartheta)) && (27)\\
& && (28)
\end{aligned}$$

which has real part

$$\Re(\varepsilon_v^n(t\cos\vartheta, t\sin\vartheta)) = t^{-1}(\cos(v\log t)\cos(n\vartheta) - \sin(v\log t)\sin(n\vartheta)). \tag{29}$$

Now let $a = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in A^+$, and consider

$$P^{+,iv}(a)\varepsilon_v^n(\sin\theta, \cos\theta) = \varepsilon_v^n(t^{-1}\cos\vartheta, t\sin\vartheta).$$

Calculation shows that $\begin{pmatrix} t^{-1}\cos\vartheta \\ t\sin\vartheta \end{pmatrix} = \begin{pmatrix} L\cos\xi \\ L\sin\xi \end{pmatrix}$ has length

$$L = \sqrt{t^{-2}\cos^2(n\vartheta) + t^2\sin^2(n\vartheta)}$$

and angle

$$\xi = \arctan(t^2\tan\vartheta).$$

Then (29) becomes

$$f(t,\vartheta) = L^{-1}\cos\left(\frac{v}{2}\log(t^{-2}2\cos^2(n\vartheta) + t^2\sin^2(n\vartheta))\right)\cos(n\xi) + \sin\left(\frac{v}{2}\log(t^{-2}\cos^2(n\vartheta) + t^2\sin^2(n\vartheta))\right)\sin(n\xi)$$

is a map $A^+ \times S^1 \to \mathbb{R}$. This is not a constructible function of ϑ, as we cannot view ϑ as restricted to any compact interval if we wish H_v^+ to be a representation, and then we cannot view cos, sin and tan as restricted-analytic. (More explicitly, sine is not a globally subanalytic function.) This shows that $P^{+,iv}(a)\varepsilon_v^n$ is generally not constructible.

2.6.2 Loss of density

The subset $G^{(0)} = KA^+K$ (or equally well for these purposes, KA^-K) is open and dense in G. $G^{(\epsilon)} := KA_\epsilon^+K$ is neither open nor dense, but for small enough ϵ includes all of G except a small neighbourhood of the identity.

3 Example for $\mathrm{SL}(2,\mathbb{R})$

We can now fix the notation from 2. The case for $\mathrm{SL}(2,\mathbb{R})$ will simplify greatly, as there will be only one positive root α to consider, and so $\Delta = \Sigma^+$, and $l = 1$. Further, as K is the circle group and in particular abelian, K-types will be one-dimensional and so the associated τ-spherical functions will take values in $\mathbb{C}$. The theory developed in section 2.2 will also simplify greatly, as $Z(\mathfrak{sl}(2,\mathbb{R})^{\mathbb{C}}) = \mathbb{C}[\Omega]$ is generated by the Casimir element, which is strongly analagous to Laplacian operators. The numbers s_i will also have an easy interpretation as roots of an indicial equation associated to series solutions for differential equations.

3.1 Coordinates on $\mathrm{SL}(2,\mathbb{R})$

[4] puts coordinates on A^+, but we can put coordinates on A^- setting for $a = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in A^-$

$$\iota(a) = \iota(\exp H) = e^{\alpha(H)} = \exp(\alpha \begin{pmatrix} \log t & 0 \\ 0 & -\log t \end{pmatrix} = \exp(-2\log t) = t^2,$$

as there is only one positive simple root α. These ι coordinates, the coordinates or [1] are both constructible, ι being a polynomial functions, on the semi-analytic domain A^-; as membership in A^- is determined by polynomial conditions and a polynomial inequality. The advantage of the ι coordinates is that they send A^-, viewed as $\exp(-\infty, 0) = (0,1)$, to itself by a polynomial, whereas coordinates on A^+ would need to map $t \mapsto t^{-2}$ in order to have precompact image.

3.2 Asymptotics of τ-spherical function for $\mathrm{SL}(2,\mathbb{R})$

3.2.1 Computing q_0

The integer $q_0 = l(n-1)$ bounds the maximum exponent of a logarithmic term in a solution function. Therefore q_0 is bounded above by one less than the maximum size of Jordan block in the fundamental matrix for the system described in §7. As $Z(\mathfrak{sl}(2,\mathbb{R})^{\mathbb{C}}) \simeq \mathbb{C}[\Omega]$ and the Casimir element Ω acts as the Laplacian, we have that $q_0 = 1$. A weaker bound, that $q_0 \leq 2$, the order of the Weyl group, is also noted in §7.

3.2.2 Bounding $\mathscr{F}$

That $Z(\mathfrak{sl}(2,\mathbb{R})^{\mathbb{C}}) \simeq \mathbb{C}[\Omega]$ shows also that we have a secon-degree differential equation in one variable. In Appendix B we have that n is the degree of the equation, so the fundamental matrix is a map

$$\Phi(z) = \begin{pmatrix} \varphi_1(z) & \varphi_2(z) \\ \varphi_1'(z) & \varphi_2'(z) \end{pmatrix} : V \subset \mathbb{C}^2 \to \mathbb{C}^2.$$

where $\{\varphi_1, \varphi_2\}$ is a basis for the space of solutions.

This corresponds to the fundamental matrix described in section 2.2 actually being a matrix. The equation, obtained after lengthy changes of coordinates carried out in [4], is

$$\frac{1}{2}D_\tau(\Omega) = \frac{d^2F}{dt^2} + (\coth t)\frac{dF}{dt} + \frac{1}{\sinh^2 t}(F(a_t)\tau_2(Y)^2 + \tau_1(Y)F(a_t)) - \frac{2\cosh t}{\sinh^2 t}\tau_1(Y)F(a_t)\tau_2(Y) = cF(a_t) \quad (30)$$

where

$$a_t = \begin{pmatrix} e^{t/2} & \\ & e^{-t/2} \end{pmatrix} = cF(a_t)$$

and

$$Y = \frac{1}{2}(e - f).$$

Here we are working with the standard basis of $\mathfrak{sl}(2,\mathbb{R})$:

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (31)$$

One then gets, by the remarks at the beginning of this section and the theory developed above, that in the coordinates ι,

$$F \circ \iota^{-1}(z) = z^{s_1} F_{1,0}(z) + z^{s_2} \log(z) F_{2,1}(z).$$

Here the s_i have an interpretation as roots of the indicial equation

$$s^2 - s = c.$$

Remark 4. These roots are generally complex, especially in cases such as admissible (in this case, unitary as representations of G) principal series representations.

Here the indicial equation is

$$w^2 - w = \frac{-v^2 - 1}{4},$$

where v is the real parameter appearing in $\mathscr{P}^{\pm,iv}$. The indicial equation has discriminant

$$1 - 4\left(\frac{v^2}{4} + \frac{1}{4}\right) = -v^2 \leq 0$$

and roots

$$\frac{1}{2} \pm i\frac{v}{2}$$

and so the indicial equation does not have real roots unless $v = 0$. The representation $\mathscr{P}^{+,0}$ is irreducible, and hence admissible, but $\mathscr{P}^{-,0}$.

Remark 5. The remark above gives an example of an admissible representation which *does* have constructible τ-spherical functions, $\mathscr{P}^{+,0}$, *provided that* $z \mapsto \sqrt{z}$ *is constructible for real* z.

Remark 6. The expansion (17) is imprecise in the sense that while the degree of the log terms appearing is bounded as described there, by $l(n-1)$, (also by the order of the Weyl group), log need not appear. Indeed, the equation above for $\mathrm{SL}(2,\mathbb{R})$ does not contain logarithmic terms unless $s_1 - s_2$ is an integer. This difference by an integer in the semi-simple parts of a family of commuting matrices corresponds, after multiplying by $-2\pi i$ and exponentiating, repeated eigenvalues of the monodromy representation matrix M. Thus the monodromy action is not semi-simple in this case.

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