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Trading Roles of Position and Direction in Kakeya Sets and Finite Field Kakeya Sets in Euclidean Kakeya Set Neighbourhoods

1 Introduction

A well-known problem linked to several areas of mathematics, e.g. incidence and algebraic geometry, is the Kakeya Conjecture, which proposes that any Euclidean Kakeya set, defined below, of $\mathbf{R}^n$ has Haussdorf and Minkowski dimensions equal to n. While several partial results exist, the full dimensional result for either definitions of dimension remains to be proved. Thus, Kakeya sets, and, in particular, their Minkowski dimensions were the focus of this USRA project. Preparation for this project involved studying past results in the Kakeya problem and various related problems. For example, geometric arguments like Bourgain's bush [1] were analyzed in addition to polynomial solutions to the finite field analogue of the Kakeya problem, by Dvir [2], and the joints problem, by Guth and Katz [3].

Having studied past approaches to Kakeya-flavoured problems, we then continued along two separate avenues of original inquiry. Firstly, as described in §2, we investigated a sequence of transformations, which, when applied to a Kakeya set E, produces a set G with an interesting property mimicking the definition of a Kakeya set, but with the roles of position and direction swapped. Secondly, we asked whether Dvir's result, provided below, for finite field Kakeya sets could be directly applied to the Euclidean setting by nesting finite field Kakeya sets within ϵ-neighbourhoods of Euclidean Kakeya sets (after some transformation). As explained further in §3, we were ultimately able to bound the Minkowski dimension of Euclidean Kakeya sets by $\geq \frac{n^2}{2n-1}$, falling short of modern bounds, though we also noted that only small portions of the each Kakeya set were utilized for this bound.

We conclude this section by providing some pertinent notational conventions, definitions, and Dvir's result. For q prime ≥ 3, $\mathbf{F}_q$ will denote a finite field of the q integers $-\frac{q-1}{2}, \dots, \frac{q-1}{2}$. Underlines subscript q will show that operations are with respect to $\mathbf{F}_q$ while operations without underline are understood with respect to $\mathbf{R}$. Additionally, the Minkowski dimension and ϵ-neighbourhood of a set A will be denoted by $\dim A$ and $N(A, \epsilon)$ respectively. Further, modulus will denote the length of vectors, cardinalities of finite sets, and Lebesgue measures of infinite sets. Finally, functions applied to sets and vectors are to be applied element-wise.

Definition 1.1. A set $E \subseteq \mathbf{R}^n$ is a Euclidean Kakeya set if for all ν in the $n-1$-dimensional unit sphere S^{n-1}, there exists $\chi \in \mathbf{R}^n$ such that $\left\{\chi + \nu\tau : \tau \in \left[-\frac{1}{2}, \frac{1}{2}\right]\right\} \subseteq E$. Unless otherwise specified, Kakeya sets are assumed to be Euclidean Kakeya sets.

Definition 1.2. A set $K \subseteq \mathbf{F}_q^n$ is a finite field (δ, γ)-Kakeya set if, for for all v in some $V \subseteq \mathbf{F}_q^n$ of size $|V| \geq \delta q^n$, there exists $x \in \mathbf{F}_q^n$ such that $\left|\left\{\underline{x + vt}_q : t \in \mathbf{F}_q\right\} \cap K\right| \geq \gamma q$.

Theorem 1.1 (Dvir [2])**.** *Let $K \subseteq \mathbf{F}_q^n$ be a finite field (δ, γ)-Kakeya set. Then $|K| \geq \binom{\lfloor q \min\{\delta,\gamma\} \rfloor + n - 1}{n}$.*

2 Trading Roles of Position and Direction in Kakeya Sets

In the the first portion of this USRA project, we introduced a sequence of transformations, which, when applied to a Kakeya set, E, produces a set G with an interesting characteristic antithesizing the definition of a Kakeya set. In particular, for some $\rho_0 > 0$ and $\rho_1 = \rho_0 + 1$, G satisfies the following: for every position $\nu \in \left[-\frac{1}{2}, \frac{1}{2}\right]^{n-1} \times \{0\}$, there is a vector $\chi \in [-(\rho_0 + \rho_1), \rho_0 + \rho_1]^{n-1} \times \{1\}$ such that $\{\nu + \chi\tau : \tau \in [\rho_0, \rho_1]\} \subseteq G$. Thus, recalling Definition 1.1, this property of G effectively trades the roles of position and direction in the line segments $\left\{\chi + \nu\tau : \tau \in \left[-\frac{1}{2}, \frac{1}{2}\right]\right\}$ from the definition of a Kakeya set.

Here, we briefly outline the sequence of transformations used to obtain G. Initially, we start with a general Kakeya set E, with the only additional requirement being boundedness. The first step involves transforming E to a set E' with more restrictive properties. Suppose E is covered by finitely-many filled cubes, centred at c_j's, all of sidelength $l_0 \in \left(0, \frac{1}{2}\right)$. Now, let C_j's be filled cubes centred at c_j's, but all of sidelength $l_1 \in \left(l_0, \frac{1}{2}\right)$, and define $E' \equiv \bigcup_j ((E \cap C_j) - c_j)$, where $-c_j$ indicates a translation by $-c_j$. It is easy to show that $\dim E' = \dim E$ and E' must now contain line segments in every direction, each of which twice intersects the surfaces of cubes with sidelengths l_0, l_1, centred at the origin. Now, a set G' is produced by applying the transformation $f(x_1, \dots, x_n) = -\frac{(x_1, \dots, x_{n-1}, 1)}{x_n}$ to a certain selection of line segments of E', the union of which we will call E''. Finally, G is obtained by a scaling of G', and letting $\rho_0 \equiv \frac{l_0}{l_1 - l_0}, \rho_1 \equiv \frac{l_1}{l_1 - l_0}$. An interesting observation is that f maps line segments as follows:

$$f\left\{(\chi_1, \dots, \chi_{n-1}, 0) + (\nu_1, \dots, \nu_{n-1}, 1)\tau : \tau \in [-a, -b]\right\} = \left\{-(\nu_1, \dots, \nu_{n-1}, 0) + (\chi_1, \dots, \chi_{n-1}, 1)\tau : \tau \in \left[\frac{1}{a}, \frac{1}{b}\right]\right\}.$$

Thus, f can be regarded as the key component of this sequence of transformations which swaps the roles of position and direction for each line segment in E'' and gives G its aforementioned property.

While G is interesting by itself due to its special property, we further motivated the study of G by showing an equivalence between proving some bound $\dim E \geq d_n$ for all Kakeya sets E and proving $\dim G \geq d_n$ for all corresponding G. To this end, we also proposed a discretized version of the problem in $\mathbf{Z}^n$, one which we optimistically hope is easier to solve. This discretized problem considers a set J_q which satisfies the following property for some q-sized interval T of positive $\mathbf{Z}$: for all $v \in \{0, \ldots, q-1\}^{n-1} \times \{0\}$, there exists $x \in \mathbf{Z}^{n-1} \times \{1\}$ such that

$$\left\{(vq + w) + xt : w \in \{0, \ldots, q-1\}^{n-1} \times \{0\}, t \in T\right\} \subseteq J_q.$$

If one were able to successfully show $|J_q| \gtrsim q^c$, then it would imply $\dim G \geq c + 1 - n$.

3 Finite Field Kakeya Sets in Euclidean Kakeya Set Neighbourhoods

In the second portion of this USRA project, we aimed to take Dvir's result, summarized in Theorem 1.1, from the finite field setting and reapply it to obtain a dimensional bound on Euclidean Kakeya sets. Here, we provide some definitions that will aid in our discussion.

Definition 3.1. For $y \in \mathbf{R}$, define $\text{nod}_q(y) \equiv \text{mod}_q(y) - \begin{cases} 0 & \text{if } \text{mod}_q(y) \leq \frac{q}{2} \\ q & \text{otherwise} \end{cases}$.

Definition 3.2. For $v, w \in \mathbf{F}_q^n \setminus \{0\}$, we say $v \parallel_q w$ if there exists $\alpha \in \mathbf{F}_q$ such that $v = \underline{\alpha w}_q$. Also, for positive $\sigma \in \mathbf{R}$, define $\Omega(v, \sigma) \equiv \{v' \in [v]_q : |v'| \leq \sigma\}$ where $[v]_q$ is the equivalence class of v under $\parallel_q$.

In the regime of q prime $\sim \epsilon^{-k}$ for some $0 < k \leq 1$, our idea was to nest the $\frac{1}{2}$-neighbourhood of some finite field (δ, γ)-Kakeya set K within the scaled and transformed $N(E, \epsilon)$ of a Euclidean Kakeya set E. Dvir's result would then bound $|K|$ from below and ultimately provide bounds for $|N(E, \epsilon)|, \dim E$. To describe the transformation on $N(E, \epsilon)$ that will allow it to house a finite field (δ, γ)-Kakeya set, it is useful to define the function $\text{nod}_q : \mathbf{R} \to \left(-\frac{q}{2}, \frac{q}{2}\right]$ as above so that the finite field line $L_{x,v} \equiv \left\{\underline{x + vt}_q : t \in \mathbf{F}_q\right\}$ can be equivalently written as $\text{nod}_q\{x + vt : t \in \mathbf{F}_q\}$. Moreover, for some large constant κ_n, since $\kappa_n \epsilon^{-1} N(E, \epsilon)$ – where multiplication indicates scaling – contains κ_n-radius, $\kappa_n \epsilon^{-1}$-length tubes in all directions, it follows that one can construct an $L_{x,v}$ with the $\frac{1}{2}$-neighbourhood of γq-many of its points within $M(E, \epsilon) \equiv \text{nod}_q(\kappa_n \epsilon^{-1} N(E, \epsilon))$, provided $|v|$ is sufficiently small compared to $\kappa \epsilon^{-1}$. Thus, to determine what δ-values can be used while guaranteeing that some finite field (δ, γ)-Kakeya set exists with its $\frac{1}{2}$-neighbourhood within $M(E, \epsilon)$, we considered the number of v that satisfies – or is $\parallel_q$ to some w that satisfies – this "sufficiently small" condition. To this end, we proved the result below.

Definition 3.3. Let $\epsilon > 0$ and constants $\kappa_n > 0$ and $0 < \gamma \leq 1$ be given. Let $v \in \mathbf{F}_q^n$. Then, v is tube-filling if there exists $\nu \in \mathbf{R}^n \setminus \{0\}$ such that the nod_q of any κ_n-radius, $\kappa_n \epsilon^{-1}$-length tube in the direction of ν contains the $\frac{1}{2}$-neighbourhood of at least γq-many points of $L_{x,v}$ for some $x \in \mathbf{F}_q^n$. For such ν, we say that v fills tubes in the direction of ν.

Theorem 3.1. *Let $\epsilon, \kappa_n, \gamma$ be as in Definition 3.3. Additionally, suppose $\kappa_n \geq \frac{\sqrt{n}+1}{2}$ and let q be prime $\lesssim \epsilon^{-1}$. Then for some γ_0, satisfying $\gamma_0 \sim \epsilon^{-1} q^{\frac{1-2n}{n}}$ for q large, the following is true. If $\gamma \geq \gamma_0$, then there exist $\sigma \sim (\epsilon \gamma q)^{-1}$ and $V \subseteq \mathbf{F}_q^n$, of size $\gtrsim (\epsilon \gamma)^{-n} q^{1-n}$, with each element $v \in V$ filling tubes in the directions of $\Omega(v, \sigma) \neq \emptyset$. If $\gamma < \gamma_0$, then there exist $\varsigma \sim q^{\frac{n-1}{n}}$ and $V \subseteq \mathbf{F}_q^n$, of size $\sim q^n$, with each element $v \in V$ filling tubes in the directions of $\Omega(v, \varsigma) \neq \emptyset$.*

Using Theorem 3.1, we found that the best bound acheivable by our method, $\dim E \geq \frac{n^2}{2n-1}$ for a general Euclidean Kakeya set E, arose from choosing $k = \frac{n}{2n-1}$ and $\delta, \gamma \sim 1$. Though this is about $\frac{1}{4}$ shy of even the $\frac{n+1}{2}$ bound from Bourgain's bush argument [1], we also noted that our method thus far has used tubes that only make up a portion H_ϵ of $\kappa_n \epsilon^{-1} N(E, \epsilon)$ when trying to obtain the bound $|\kappa_n \epsilon^{-1} N(E, \epsilon)| \gtrsim \epsilon^{-\frac{n^2}{2n-1}}$ that ultimately led to our bound on $\dim E$. Thus, perhaps our result is most strongly stated as follows:

Theorem 3.2. *There exists a positive constant $\kappa_n \in \mathbf{R}$, possibly dependent on n only, such that the following is true. Let $\epsilon > 0$ and q be prime $\sim \epsilon^{-\frac{n}{2n-1}}$. Let $W \subseteq \mathbf{F}_q^n \setminus \{0\}$ contain an element from each $\Omega(v, \varsigma)$ for $v \in V$, where ς, V are as guaranteed by Theorem 3.1 for the $1 \sim \gamma < \gamma_0 \sim 1$ case. If $H_\epsilon \subseteq \mathbf{R}^n$ contains κ_n-radius, $\kappa_n \epsilon^{-1}$-length tubes in directions of all elements of W, then $|H_\epsilon| \gtrsim \epsilon^{-\frac{n^2}{2n-1}}$.*

In particular, observe that H_ϵ can contain as few as $\sim q^{n-1} \sim \epsilon^{-\frac{n(n-1)}{2n-1}}$-many tubes, whereas $\kappa_n \epsilon^{-1} N(E, \epsilon)$ necessarily contains tubes in all directions. Therefore, we can write $\kappa_n \epsilon^{-1} N(E, \epsilon) \supseteq \bigcup_j H_{\epsilon,j}$ for many $H_{\epsilon,j}$'s, each of which is a rotation of some valid choice of H_ϵ and necessarily satisfies $|H_{\epsilon,j}| \gtrsim \epsilon^{-\frac{n^2}{2n-1}}$. To improve our current dimensional bound on E, the challenge would then be to show some degree of disjointness between $H_{\epsilon,j}$'s and hence improve the crude bound $\left|\bigcup_j H_{\epsilon,j}\right| \geq |H_{\epsilon,j}|$.

References

[1] J. Bourgain, *Besicovitch type maximal operators and applications to Fourier analysis*, Geom. Funct. Anal. **1** (1991), no. 2, 147-187.

[2] Z. Dvir, *On the size of Kakeya sets in finite fields*, J. Amer. Math. Soc. **22** (2009), no. 4, 1093-1097.

[3] L. Guth and N. Katz, *Algebraic methods in discrete analogs of the Kakeya problem*, Adv. Math. **225** (2010), no. 5, 2828-2839.