Summer 2015 - NSERC USRA Report
Sets Avoiding Images of a Given Sequence

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This summer I worked with Dr. Malabika Pramanik on the following conjecture of Erdős. Define a set $A \subset \mathbb{R}$ to be *universal* if every set $E \subset \mathbb{R}$ of positive Lebesgue measure contains an affine copy of A; if $\exists x, t \in \mathbb{R}$, $t \neq 0$, $x + tA = \{x + ta : a \in A\} \subset E$.

Conjecture 0.1 (Erdős) Every infinite set is non-universal.

Note that every finite set A is universal. For instance, consider $A = \{-1, 0, 1\}$ and suppose E has positive Lebesgue measure and contains no affine copy of A. By the Lebesgue density theorem, some point $x \in E$ has density one; however, $\forall \epsilon > 0$, to each point of $(x - \epsilon, x)$ corresponds one of $(x, x + \epsilon)$ such that one must be excluded, so the density is at most $1/2$, a contradiction. This argument easily generalizes.

Returning to Erdős' conjecture, because of the scale and translation-invariance of the problem, one typically considers A to be a positive, decreasing sequence $\{a_n\}$ converging to 0, although even for some uncountable sets A the conjecture has not been established. Falconer [1] has confirmed non-universality in the case where $\{a_n\}$ does not decrease too quickly, using a Cantor-type construction.

Theorem 1 (Falconer, 1984) Suppose $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. Then A is non-universal.

Falconer's theorem establishes the conjecture where $\{a_n\}$ decays polynomially, and Kolountzakis [2] generalizes this argument to where $\delta(n) = \min_{i<n} a_i - a_{i+1}$ has $-\log \delta(n) \in o(n)$, which is the case with $A = \{2^{-n^\alpha}\} + \{2^{-n^\alpha}\}$, $0 < \alpha < 2$. However, the case $A = \{2^{-n}\}$ is still open. We considered this case, drawing on a probabilistic construction of Kolountzakis:

Theorem 2 (Kolountzakis, 1997) There exists a set E of positive measure such that the two-dimensional Lebesgue measure of the bad (x, t)-pairs $m\{(x, t) : x + tA \subset E\} = 0$.

One can restrict the scaling parameter t to an interval $[\alpha, \beta]$ and intersect the countably many corresponding sets $E \subset [0, 1]$, provided their measure can be made arbitrarily close to 1. Fix probabilities $\{p_k\}$ with $\prod_k p_k$ arbitrarily close to 1 and $\prod_k p_k^k = 0$. Letting $m_k \in \mathbb{N}$ large enough so that $\frac{1}{m_k} < \alpha \min_{i<k} a_{i+1} - a_i$, divide $[0, 1]$ into m_k equal

pieces and select each independently randomly with probability p_k. Let E_k be the union of the selected intervals and $E = \cap_k E_k$. For $x \in [0,1]$, if $x \in E$ then $x \in$ each E_k; the probability of this is $\prod_k p_k$. On the other hand, if for some $x \in \mathbb{R}$, $t \in [\alpha, \beta]$ nonzero, $x + tA \subset E$, then by the choice of the m_k, at each step k distinct intervals are required; this occurs with probability $\prod_k p_k^k = 0$. Therefore, one can find a suitable E.

Kolountzakis also showed that the exceptional pairs project to a null set on the t-axis. Note that projection to a null set on the x-axis would be sufficient for non-universality:

Lemma 0.2 Suppose $\exists E \subset \mathbb{R}$ with $m(E) > 0$ and $m(P) = 0$ where $P = \{x : \exists t : x + tA \subset E\}$. Then A is non-universal.

Proof: P has an open cover Q with $F = E \setminus Q$ having positive measure. If for some x, t, $x + tA \subset F \subset E$, then by definition $x \in P \subset Q$. But Q is open so $\exists a \in A$ with $x + ta \in Q$, contradicting $x + tA \subset F$. ■

Therefore, for $A = \{2^{-n}\}_{n=0}^{\infty}$, we divide $(0,1)$ into intervals $I_{a_1} = (2^{-a_1}, 2^{-a_1+1})$, selecting each with probability p_1. Then divide each I_{a_1} into intervals $I_{a_1,a_2} = (2^{-a_1} + 2^{-(a_1+a_2)}, 2^{-a_1} + 2^{-(a_1+a_2)+1})$, selecting with probability p_2 and so on, constructing E as above. We aim to show that $\mathrm{Exp}(m(\{x : \exists s \in [1,2] : x + s2^{-k}A \subset E\})) = 0\ \forall k \in \mathbb{N}$.

Fix $x \in (0,1)$ and $k \in \mathbb{N}$; $x \in I_{a_1,\ldots,a_n,\ldots}$ and let n be such that $a_1 + \ldots + a_n \geq k$. We show that if x is bad, then *all* subsequent subintervals of $I_{a_1,\ldots,a_n}$ must have been included. Note that to calculate the appropriate expectation, the required subintervals must be independent of the scaling.

Lemma 0.3 Suppose $\exists s \in [1,2]$ with $x + s2^{-k}A \subset E$. Then $\forall j \leq a_{n+1}$, $I_{a_1,\ldots,a_n,j} \subset E_{n+1}$. (This happens with probability $p_{n+1}^{a_{n+1}}$.)

Unfortunately, this lemma provides scant information when a_{n+1} is small, which will usually be the case. I managed to show that it is not always the case:

Lemma 0.4 Let $N(n) \in O(\log(n \log n))$. Then for almost every $x \in (0,1)$, $a_n > N(n)$ for infinitely many n.

Still, since this lemma extracts only a subsequence, we can not ensure that $\prod_n p_{n_k}^{a_{n_k}} = 0$ for every subsequence while $\prod_n p_n$ is arbitrarily close to 1. Hopefully, one can refine the probability estimate, possibly by considering required cousin intervals.

References

[1] Falconer, K. J. *On a problem of Erdős on sequences and measurable sets.* Proc. Amer. Math. Soc. 90 (1984), no. 1, 7778. (Reviewer: V. Losert) 28A75 (11K55)

[2] Kolountzakis, Mihail N. *Infinite patterns that can be avoided by measure.* Bull. London Math. Soc. 29 (1997), no. 4, 415424. (Reviewer: M. Laczkovich) 28A05