Left-Orderability, Branched Covers and Representations

NSERC USRA Report - Summer 2015

Supervisors: Dr. Steven Boyer (UQàM) and Dr. Dale Rolfsen (UBC)

Darrick Lee

1 Introduction

This summer, I was working with Professors Steven Boyer and Dale Rolfsen on a project in the field of algebraic topology on the topic of left-orderability of fundamental groups of 3-manifolds. The fundamental group $\pi_1(X)$ is a group that consists of all possible closed paths in the space X up to homotopy equivalence (we consider two loops that can be continuously deformed into one another to be the same). The fundamental group is important because it is a topological invariant - in fact, if two spaces are homotopy equivalent, then their fundamental groups are the same. Thus, we can study fundamental groups to better understand the corresponding topological spaces. Recently, Professor Boyer along with his collaborators have conjectured a connection between a topological property of 3-manifolds and an algebraic property of the corresponding fundamental group.

Conjecture 1 ([1])**.** *An irreducible rational homology 3-sphere is an L-space if and only if its fundamental group is not left-orderable.*

An L-space is a class of rational homology 3-spheres defined by Oszvath and Szabo [3] which requires very technical concepts to properly define. On the other hand, left-orderability is a concept which has a simple definition.

Definition 1. *A group G is **left-orderable** if there exists a strict total order $<$ such that for all $f, g, h \in G$, the expression $g < h$ implies $fg < fh$.*

Therefore, this conjecture is important because it relates a fairly technical concept to one which is much simpler. My project was to look at spaces with left-orderable fundamental groups, which could be examples to this conjecture. This began with a study of branched covers of knots.

2 Left Orderability and Branched Covers

Definition 2. *Suppose $B_M \subset M$ and $B_X \subset X$. A **branched cover** of a space X branched over B_X is a space M and a map $p : M \to X$ such that for every $x \in X - B_X$, there exists an open neighbourhood of N such that the preimage $p^{-1}(N) \subseteq M - B_M$ is a disjoint union of open sets, each homeomorphic to N. The subsets B_M and B_X are the **branch sets** of X.*

An example will be given to demonstrate this concept. Consider the unit disc in $\mathbb{C}$, denoted by $D = \{z \in \mathbb{C} : |z| \leq 1\}$. It turns out that the unit disc can be a branched covering of itself, so both $X = D$ and $M = D$ and we take $p(z) = z^2$, with branch sets $B_X = B_M = \{0\}$. The details of the definition can be confirmed by using Fig. 1. The branched covering of a space M can be though of as a space M that is n copies of X except in the branch sets (in this example, M is a twofold branched cover).

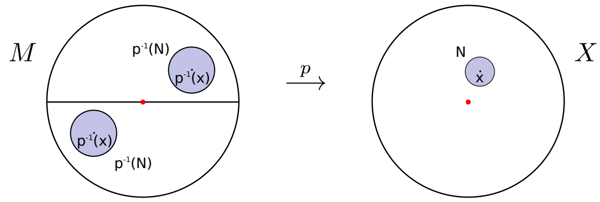

Figure 1: Example of a branched cover of the unit disc.

Definition 3. *A* ***knot*** *K is a subset of $S^3 = \{(x, y, z, t) \in \mathbb{R}^4 \,|\, x^2 + y^2 + z^2 + t^2 = 1\}$ that is homeomorphic to a circle $S^1 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$. A* ***branched cover of a knot*** *K is a branched covering of S^3 where the branch set is the knot K.*

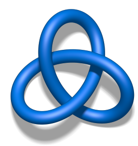

Figure 2: The trefoil knot, an example of a knot.

Hu [2] proved a sufficient condition for the cyclic branched cover of a knot to be left-orderable. She applied this condition to show that a large class of two-bridge knots had only finitely many branched covers which are not left-orderable. Tran [6] also applied this condition to another class of two-bridge knots called double twist knots to show they had the same property. Hu's condition says that if there exists a real parabolic representation of the knot group, then only finitely many branched covers are not left-orderable. Thus, my project was to investigate real parabolic representations of yet another class of two bridge knots.

3 Representations of Knot Groups

Definition 4. *Suppose K is a knot. A* ***real representation*** *of the knot group $\pi_1(S^3 - K)$ is a homomorphism $\rho : \pi_1(S^3 - K) \to SL_2(\mathbb{R})$, where $SL_2(\mathbb{R})$ is the set of 2×2 real matrices with determinant 1. A representation is called* ***parabolic*** *if some set of meridional generators of $\pi_1(S^3 - K)$ are mapped to elements of trace 2.*

In [4], Riley showed that the existence of real parabolic representations corresponded to the existence of real roots of a polynomial, now called the Riley polynomial. In addition, he conjectured a relationship between the number of such representations and the signature of the knot.

Conjecture 2 ([4])**.** *Suppose K is a two-bridge knot. Then $\pi_1(X_K)$ has at least $\frac{1}{2}|\sigma(K)|$ real parabolic representations, where $\sigma(K)$ is the signature of the knot.*

For my project, I was able to verify Riley's conjecture for two classes of two bridge knots (in Schubert's normal form notation [5]): the $(4mn + 1, n)$ and $(4mn + 2n - 1, n)$ knots, where $m > 0$, n is odd and $n > 1$. It turns out the signature of these knots is nonzero, so there exists at least one representation, so Hu's condition can be applied to show that only finitely many branched covers of these knots are not left-orderable.

References

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