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PATTERNS IN SPARSE SETS

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1. Introduction

This summer I worked with Prof. Malabika Pramanik on a problem of Erdős posed many years ago: Given any infinite set A, does there exist a measurable set of positive measure E which avoids all affine copies of A (sets of the form $x + tA$)? While this naturally arising question was asked over 40 years ago, there are very few results concerning it. Prof. Pramanik and I spent the summer reviewing the literature and examining a special case of the question, namely when the infinite set A is the geometric series $\{2^{-n}\}_{n\in\mathbb{N}}$.

2. History

As previously stated, the question was first asked by Paul Erdős many years ago, and recorded in [2] in 1981.

In 1983 Komjáth proved in [5] that for every subset A of $[0, 1]$ there exists another subset of $[0, 1]$ with measure arbitrarily close to 1 which avoids all translates of A.

Falconer showed that if our set $A = \{a_i\}_{i\in\mathbb{N}}$ is a sequence decreasing to 0 such that $\lim \frac{a_{n+1}}{a_1} = 1$, then there exists another set avoiding all affine copies of it.

Bourgain tackled the problem from a different perspective, looking at its higher dimensional analogue. In [1] he shows that sets of the type $S_1 + S_2 + S_3$ where the S_i are infinite, as well as some sets like $\{2^{-n}\} + \{2^{-n}\}$ both answer no to Erdős' question. His paper also quantifies this property in terms of an integral.

There is also an interesting paper [4] by Kolountzakis eliminating some further slowly decreasing sets, as well as showing that for any set there are big sets for which the measure of the possible scalings t is zero. The results here imply several of the earlier results.

There have been many different interesting approaches as well, such as [3] by Humke and Laczkovich, which translate the problem into a finite combinatorial one.

3. Our work

Prof. Pramanik and I wished to expand on the result of Kolountzakis in [4], namely Theorem 3. We try to improve the argument for the special case of the infinite set A being $\{2^{-n}\}_{n\in\mathbb{N}}$.

Below I outline our attempt at constructing a set of positive measure which avoids all affine copies of $\{2^{-n}\}_{n\in\mathbb{N}}$, and our progress in checking whether this set works.

3.1. **Constructing the set.** Keeping the same notation as in Kolountzakis' paper, let $A(N) = \{a_0, \dots a_N\}$. We fix the scaling factor t to be in $[2^{-l}, 2^{-l+1}]$. We wish to find an $E \subseteq [0,1]$ of measure arbitrarily close to 1 depending on N such that $\mu\left(\{x : \exists t \in [2^{-l}, 2^{-l+1}] \text{ st. } (x + tA(N)) \subseteq E\}\right)$ goes to 0 as N goes to ∞. This is sufficient; we can then intersect countably many of the E very close to measure 1 for each scaling factor range, and then remove a small covering of the bad x.

We define E by

$$1_E = \sum_{i=0}^{2^{N+l}-1} X_i 1_{[\frac{i}{2^{N+l}}, \frac{i+1}{2^{N+l}} - \frac{\varepsilon}{2^{N+l}}]},$$

where the $X_i \in \{0,1\}$ are independent indicator random variables with a fixed expected value $p \to 1$ such that $p^n \to 0$, and fixing $\varepsilon = 2^{-R}$.

Similarly, define $\tilde{E}$ to be

$$1_{\tilde{E}} = \sum_{i=0}^{2^{N+l}-1} X_i 1_{[\frac{i}{2^{N+l}}, \frac{i+1}{2^{N+l}}]}.$$

3.2. **Discretizing the set.** We repeat some results originally in [4] here. We claim that it suffices to consider a finite number of scaling parameters, that is there exists a finite set of real numbers t_k such that

$$\{x : \exists t \in [2^{-l}, 2^{-l+1}] \text{ st. } x + tA \subseteq E\} \subseteq \bigcup_k \{x : x + t_k A \subseteq \tilde{E}\}.$$

To see this, for an x in the first set take $\frac{1}{2^l}\left(1 + \frac{k\varepsilon}{2^N}\right)$ with the smallest integer k which makes it larger than x. Each point of A then moves forward less than $\frac{\varepsilon}{2^{l+N}}$ when shifted by t_k, and is thus in the same segment as when shifted by some corresponding t.

We want to inspect the set of bad x above. Our eventual goal is to shrink this as much as possible, and then remove them from our set. Consider

$$1_{\tilde{E}-t_k a_j}(x) = \sum_{i=0}^{2^{N+l}-1} X_i 1_{[\frac{i}{2^{N+l}}, \frac{i+1}{2^{N+l}}]}(x + t_k a_j) = \sum_{i=0}^{2^{N+l}-1} X_i 1_{[\frac{i}{2^{N+l}} - t_k a_j, \frac{i+1}{2^{N+l}} - t_k a_j]}(x).$$

Since we know $t_k = \frac{1}{2^l}(1 + \frac{k}{2^{N+R}}) = \frac{2^{N+R}+k}{2^{N+R+l}}$, and $a_{N-j} = \frac{2^j}{2^N}$, we can write

$$1_{\tilde{E}-t_k a_j}(x) = \sum_{i=0}^{2^{N+l}-1} X_i 1_{[\frac{i2^{N+R} - 2^{N+j+R} - k2^j}{2^{2N+l+R}}, \frac{(i+1)2^{N+R} - 2^{N+j+R} - k2^j}{2^{2N+l+R}}]}$$

Now instead of summing intervals of length $\frac{1}{2^{N+l}}$, we sum intervals of size $\frac{1}{2^{2N+l+R}}$ to get

$$\sum_{m=0}^{2^{2N+l+R}-1} Y_m 1_{[\frac{m - 2^{N+j+R} - k2^j}{2^{2N+l+R}}, \frac{m+1-2^{N+j+R} - k2^j}{2^{2N+l+R}}]},$$

where the Y_m are defined to be the corresponding X_i, so $Y_m = X_{\lfloor \frac{m}{2^{N+R}} \rfloor}$.

The above sum expresses the bad x which only shift one a_j, that is $\{x : \exists t_k \text{ st. } x + t_k a_j \in E\}$. So if we want all a_j, we look at the intersection of these sets for each $j = 0, 1, \dots, N$, which yields a product

$$\prod_j \sum_{m=0}^{2^{2N+l+R}-1} Y_m 1_{[\frac{m - 2^{N+j+R} - k2^j}{2^{2N+l+R}}, \frac{m+1-2^{N+j+R} - k2^j}{2^{2N+l+R}}]},$$

and all that survives is of the form

$$\sum_a (\prod_p Y_p) 1_{[\frac{a}{2^{2N+l+R}}, \frac{a+1}{2^{2N+l+R}}]}.$$

In the above sum, a runs between $2^{2N+l+R}[\max_j -t_k a_j, \min_j 1 - t_k a_j] = [-2^{N+R} - k, 2^{2N+l+R} - 2^{2N+R} - k2^N]$.

So the above sum more precisely is

$$\sum_{a=-2^{N+R}-k}^{2^{2N+l+R}-2^{2N+R}-k2^N-1} \left(\prod_{j=0}^{N} Y_{a+2^{N+j+R}+k2^j-2^{N+R}} \right) 1_{[\frac{a}{2^{2N+l+R}}, \frac{a+1}{2^{2N+l+R}}]}.$$

3.3. **Testing the set.** What we wanted to produce was a set arbitrarily close to measure 1 whose measure of bad translates x went to 0 as N increased. Some parameters were not explicitly defined in the framework of the set, such as the thickening (R) and the expected value of the X_i (p).

The main question at hand is whether we can pick a p such that the set of bad x decrease. Extensive numerical testing shows that if p were allowed to be constant (causing our argument to fail elsewhere), the set of bad x does go to 0. However for larger p which satisfy our conditions, such as $p = 1 - \frac{1}{\sqrt{n}}$, the results are inconclusive. We plan on running further numerical tests to get a better idea of how this set behaves with various values for p.

3.4. **Time limitations and future plans.** Due to summer being so short, Prof. Pramanik and I are still looking at the problem, with no major results yet. However, two courses of action are most likely: either the numerical testing indicates that a certain p should work, in which case we try to proceed with a proof, or no p seems to do the job. In the latter case, we do not have much hope for our approach to work, and after checking whether special properties of $\frac{1}{2^i}$ can help, we would consider a different approach to the problem.

References

[1] J. Bourgain. Construction of sets of positive measure not containing an affine image of a given infinite structure. *Israel Journal of Mathematics*, 60:333–344, 1987. 10.1007/BF02780397.

[2] P. Erdős. *My Scottish book "problems"*. Birkhäuser. 1981.

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[5] P. Komjáth. Large sets not containing images of a given sequence. *Can. Math. Bull.*, 26:41–43, 1983.