Adapting the Hermite Cubic Collocation Scheme for Asymmetric Hydraulic Fractures

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Over this summer, I had the great privilege of working with Dr. Anthony Peirce in the problem area of mathematically modelling hydraulic fractures. The project largely centered on modifying computer code based on Dr. Peirce's Hermite cubic collocation scheme [1].

Hydraulic fractures are a class of fractures that propagate underground due to the injection of a viscous fluid that exerts pressure on the surrounding rock. The oil and gas industry uses hydraulic fractures (in a process colloquialized as "fracking") to create permeable pathways in particularly impermeable rock, such as shale, in order to improve the recovery of hydrocarbons. Unfortunately, the use of hydraulic fractures in this way is a double-edged sword: if the fractures propagate beyond their intended domain, there can be a substantial loss of hydrocarbons and safety consequences, such as toxic chemicals leeching into water supplies. These circumstances give rise to a need for mathematical study of hydraulic fractures to provide accurate models predicting their movement, such as the one I researched.

The tip of an advancing one-dimensional (growing along only one axis) hydraulic fracture can be represented schematically as in Figure 1.

Here x represents position along the fracture's axis, t is time, $w(x,t)$ is the fracture width, $q(x,t)$ is the flux, $p_f(x,t)$ is the fluid pressure, $p(x,t)$ is the overall pressure, $\ell(t)$ is the fracture length, and $x_1(t)$ and $x_2(t)$ are the left and right tips of the fracture, respectively. After choosing a characteristic length ℓ_*, time t_*, and width w_*, the characteristic pressure $p_* = \frac{w_* E}{\ell_*(1-\nu^2)}$ and flux $q_* = \frac{w_*^3 p_*}{12\mu\ell_*}$ are defined, where μ is the viscosity of the fluid and E and ν are the Young's modulus and Poisson's ratio, respectively, of the surrounding rock. The variables can then be rescaled as in Eqs. 1.

$$\begin{aligned} t = t_*\tau, \qquad x_1 = \ell_*\chi_1(\tau), \qquad x_2 = \ell_*\chi_2(\tau), \\ \gamma(\tau) = \chi_2(\tau) - \chi_1(\tau), \qquad x = \ell_*\left(\chi_1(\tau) + \gamma(\tau)\xi\right), \\ w = w_*\Omega(\xi,\tau), \qquad p = p_*\Pi(\xi,\tau), \qquad q = q_*\Psi(\xi,\tau). \end{aligned} \tag{1}$$

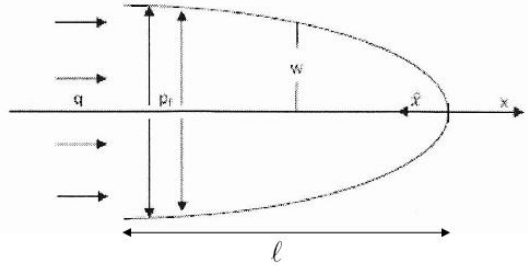

Figure 1: The various variables describing the condition of the fracture [3].

If Q_0 is the injection rate of fluid into the crack, we define the dimensionless quantities $\mathcal{G}_m = \frac{12\mu\ell_*^3(1-\nu^2)}{w_*^3 E t_*}$ and $\mathcal{G}_v = \frac{Q_0 t_*}{\ell_* w_*}$. Let $\xi_0(\tau)$ be the ξ-coordinate of the injection site. The governing equations of the hydraulic fracture relating its variables then take the form in the following equations. Eq. 2 is the elasticity equation, Eq. 3 is Poiseuille's law, Eq. 4 is the conservation law, and Eq. 5 represents global volume balance. More introduction to these equations can be found in Dr. Peirce's papers [1, 2].

$$\Pi = -\frac{1}{4\pi\gamma} \int_0^1 \frac{\Omega}{(\xi' - \xi)^2}\, d\xi'. \tag{2}$$

$$\Psi = -\frac{\Omega^3}{\gamma} \frac{\partial \Pi_f}{\partial \xi}. \tag{3}$$

$$\gamma \frac{\partial \Omega}{\partial \tau} - \left(\frac{d\chi_1}{d\tau} + \frac{d\gamma}{d\tau}\xi \right) \frac{\partial \Omega}{\partial \xi} + \frac{1}{\mathcal{G}_m} \frac{\partial \Psi}{\partial \xi} = \mathcal{G}_v \delta(\xi - \xi_0). \tag{4}$$

$$\int_0^1 \Omega\, d\xi = \mathcal{G}_v \tau. \tag{5}$$

If we create a mesh of points $\{\xi_1, \xi_2, \ldots, \xi_N, \xi_{N+1}\}$ spaced out along the length of the fracture, these equations give us a means of solving numerically for the fracture variables by time stepping and using Newton's method to converge to the correct solution. Our method uses Hermite cubic basis functions to give a third-degree polynomial level of accuracy when interpolating the fracture width between mesh points. For this reason, both Ω and its spatial derivative are tracked at each mesh point, and the elasticity equation is made to hold at the points halfway between meshpoints to handle the extra degrees of freedom. More explicitly, the algorithm solves for the variables $\Omega_2, \Omega_3, \ldots, \Omega_N, \Omega_2', \ldots, \Omega_N', \Pi_2, \ldots, \Pi_N, \Psi_2, \ldots, \Psi_N, \chi_1$, and χ_2 at each time step. Note that boundary conditions determine the fracture variables at the endpoints.

This algorithm was originally developed to model a symmetric fracture under no stress (pressure) gradient in both viscosity- and toughness-dominated regimes [1]. I helped to extend the applicability of this code to enforce no symmetry assumption and allow the addition of a stress gradient, which serves to make the crack propagate chiefly in one direction. The output of this script was then compared to that of an earlier method for modelling the fracture, dubbed ILSA (implicit level set algorithm) [2]. By contrast, this algorithm assumes a piecewise constant width function instead of using cubic basic functions.

As can be seen in Figure 2, the fracture shapes produced by the different algorithms is similar but not identical. Through an assortment of tests, we were able to verify that the shape of the fracture generated by the Hermite code was highly dependent on how large the tip sizes—in other words, $\xi_2 - \xi_1$ and $\xi_{N+1} - \xi_N$, and particularly the latter—were. Because of the tip asymptotic behaviour, special relations hold close to the tip of the fracture [1], and so if these sizes were decreased, the fracture grew laterally faster and more closely resembled the ILSA fracture. Our hypothesis is that if the mesh could be constantly shifted to decrease the tip size as time elapsed, the two fractures would converge to one another.

Though I had enrolled in two computer science courses before this summer's work, I had little experience in reading or writing long scripts, particularly those with a mathematical basis. Nevertheless, the nature of this project allowed for a practical introduction into computational methods. The considerable use of the MATLAB programming language over the summer gave me a good deal of experience that will aid me in future work.

Dr. Peirce's 2010 summer student Oren Rippel and 2008 student Shira Daltrop also worked on problems related to hydraulic fractures; for more information, see their reports.

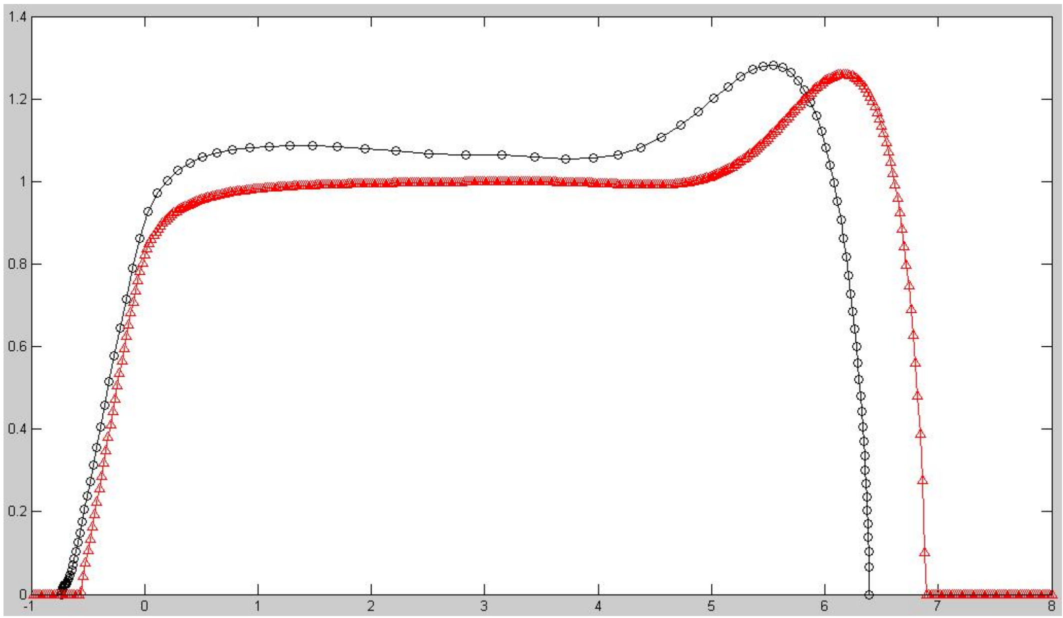

Figure 2: The fracture outline produced by the Hermite code (black) and the ILSA code (red).

References

[1] A. Peirce, A Hermite cubic collocation scheme for plane strain hydraulic fractures, *Comput. Meth. Appl. Mech. Eng.* **199** (2010), 1949-1962.

[2] A. Peirce and E. Detournay, An implicit level set method for modeling hydraulically driven fractures, *Comput. Meth. Appl. Mech. Eng.* **197** (2008), 2858-2885.

[3] J. Ernest, Numerical Analysis of Hydraulic Fractures, B.Sc. Thesis in Mathematics, 2010, 23 pp.