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NSERC USRA Report

Vector Bundles Over Toric Varieties

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$$h^0(\mathbb{P}^1, \mathrm{End}(\mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(d_n))) = h^0(\mathbb{P}^1, \bigoplus_{1\le i,j\le n} \mathcal{O}_{\mathbb{P}^1}(-d_i) \otimes \mathcal{O}_{\mathbb{P}^1}(d_j)) = h^0(\mathbb{P}^1, \bigoplus_{1\le i,j\le n} \mathcal{O}_{\mathbb{P}^1}(-d_i + d_j))$$
$$= \sum_{\substack{1\le i,j\le n \\ -d_i+d_j\le 0}} (-d_i + d_j) + 1 \ge \binom{n+1}{2}$$

Q1. Does there exists a non-affine toric variety without a non-trivial (equivariant) vector bundle?

more genereally, can you find a complete variety without a non-trivial vector bundle?
Also you can consider similiar type of questions in your favorite category.

If the toric variety is smooth then by computing equivariant K-theory, we can show that there should exist a non-trivial equivariant vector bundle.

$K_0(X(\Delta), T) \otimes_{RT} \mathbb{Z} = K_0(X(\Delta)) = \mathbb{Z}^{\#\text{ maximal cones of }\Delta} \neq \mathbb{Z}$ $\Longrightarrow$ There exist a non-trivial vector bundle, since the K-theory is not $\mathbb{Z}$ for fans with more than 1 maximal cones.

in algebraic category

in topological category

$Vect(\mathbb{P}^1) = \{E \to \mathbb{P}^1\} \Longleftrightarrow \underset{\text{as a monoid ring}}{\mathbb{Z}^{\ge 0}[\mathrm{Pic}(\mathbb{P}^1)]} = \mathbb{Z}^{\ge 0}[\mathbb{Z}] = \{\mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(d_m)\}_{d_1,\ldots,d_m \in \mathbb{Z}}$

$Vect(\mathbb{P}^1) = \{E \to \mathbb{P}^1\} \Longleftrightarrow \mathbb{Z} \oplus \mathbb{Z} = \{(rank(E), deg(E) = c_1(det(E)))\}$

$$h^0(\mathbb{P}^1, \mathrm{End}(\mathcal{O}_{\mathbb{P}^1}(d) \oplus \mathcal{O}_{\mathbb{P}^1}(0)^{\otimes n-1})) = \begin{cases} 1 + (n-1)^2 + (n-1)(|d|+1) & \text{if } d \neq 0 \\ n^2 & \text{if } d = 0 \end{cases}$$

Over $\mathbb{P}^1$ the only stable bundles are the line bundles.

$$H^0(\mathbb{P}^1, \mathrm{End}(\mathcal{O}_{\mathbb{P}^1}(d) \oplus \mathcal{O}_{\mathbb{P}^1}(0)^{\otimes n-1})) = \left\{ \begin{bmatrix} * & \cdots & & & \\ & * & * & \cdots & * \\ & * & * & \cdots & * \\ & \vdots & \vdots & \ddots & \vdots \\ & * & * & \cdots & * \end{bmatrix} \right\}$$

if $d \neq 0$

M_1

M_n

polynomials of degree $\le d$ in $k[t]$

Exercise. Classify all vector bundles over $\mathbb{A}^1$ and their automorphism groups.
Exercise. Classify all vector bundles over $\mathbb{P}^1$ and their automorphism groups.
Exercise. Construct a collection of vector bundles that would generate the K-theory for a smooth toric variety.
Exercise. Classify all vector bundles over a toric variety with only two maximal cones.
Exercise. Characterize line bundles over toric varieties.
Exercise. Find the automorphism group of the trivial bundle.
Exercise. Topological vector bundles of rank higher than the dimension of the base space split.
Exercise. Topological vector bundles of rank higher than the dimension of the base space are not simple (and thus not stable).
Exercise. Decomposable (splittable) vector bundles are not simple (and thus not stable, also in th TOP they have split with trivial line bundle as a component) .

Adam's Operations

Let $Q_k(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$ be the k^{th}-newton polynomial, i.e. if $e_i = \sum_{1\le j_1 \le j_1 \le \ldots j_i \le n} x_{j_1} \cdots x_{j_i}$ then $Q_k(e_1, \ldots, e_n) = x_1^k + \cdots + x_n^k$.
Then the k^{th}-adam's operation is defined for a vector bundle $E \to X$ by

$$\lambda^k(E) = Q_k(E^{\wedge^1}, \ldots, E^{\wedge^{rank(E)}})$$

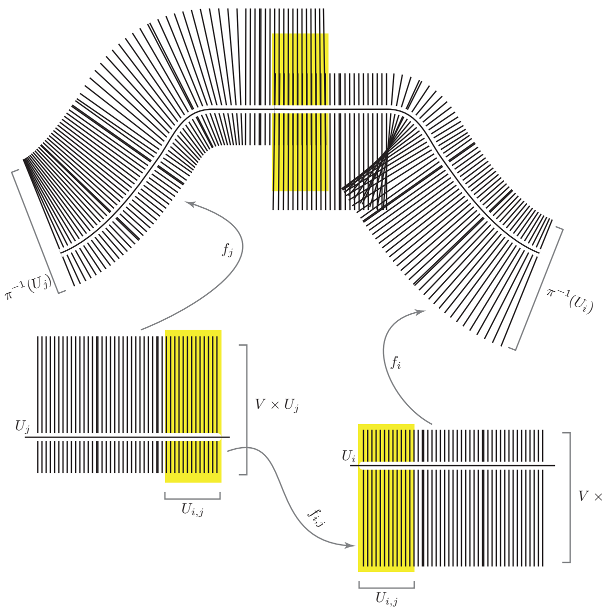
Note. To compute the k^{th}-operation we only need to wedge k-times. The above operation defines a class in the K-theory and also extends to virtual bundles as well.
Example. $\lambda^2(E) = E^{\otimes 2} - 2E^{\wedge^2}$
Exercise. If $L_i \to X$'s are line bundles then $\lambda^k(L_1 \oplus \cdots \oplus L_m) = L_1^{\otimes k} \oplus \cdots \oplus L_m^{\otimes k}$.
Exercise. Extend the above construction to virtual bundles, i.e. all elements of the K-theory.
Exercise. $\lambda^n \circ \lambda^m = \lambda^{nm}$
Exercise. $\lambda^k(E_1 \oplus E_2) = \lambda^k(E_1) + \lambda^k(E_2)$ where $E_i \to X$'s are vector bundles.
Exercise. The induced adam's operations on $\mathbb{Z}$ by the composition oth e rank map are just identities, i.e. adam's operations preserve the virtual rank.
Exercise. s-Eigenvectors of an adam's operation λ^k should have virtual dimension 0 when $s \neq 1$.

There exists a non-trivial line bundle over a toric 3-fold iff there exists one for each toric 3-fold corresponding to the stars of the rays of the fan of the original 3-fold. More generally to find non-trivial line bundles one should probably look at stars of the codim-2 cones.

In the case of a toric 3-fold if the rays belong to more than 3 codim-1 cones (2-cones) or if the normals of the codim-1 cones do not generate the dual space, then there exist a non-trivial line bundle which also represents a non-trivial class in the K-theory.
(rays + 2-cones)

$$\mathrm{rank}(\mathrm{Pic}(X(\Delta))) \ge \#\Delta^2 - \#\Delta^1$$

Construction of Vector Bundles



Exercise. Prove for every fan there exist a branched covering of the fan such that the corresponding toric variety has a non-trivial line bundle.
Exercise. Classify toric varieties up to algebraic cobordism. Do the same thing for (equivariant)vector bundles over toric varieties.
Exercise. Find all (equivariant) automorphisms of a toric variety. $\mathrm{Aut}_T(X(\Delta)) \preceq \mathrm{Aut}(\mathrm{char}(T)^\vee) \cong SL_{dim T}(\mathbb{Z})$

Vector bundles $\pi : E \to B$ locally look like a product of a vector space with a base space, i.e. we can cover B with collection of open subsets $\{U_i\}_{i\in I}$ such that for every $i \in I$ we have $\pi^{-1}(U_i) \cong V \times U_i$ for a fixed vector space V. (The above definition or a slightly modified version but with the same intution should give the desired structured object in your category of interest.) Also given the local data plus cocycle conditions we can form a bundle.

open set

$$B = \bigcup_{i\in I} U_i \ , \ U_A = \bigcap_{i\in A} U_i$$

$$V \times U_i \xrightarrow{f_i(\cong)} \pi^{-1}(U_i)$$

$V \times U_i \xrightarrow{f_i(\cong)} \pi^{-1}(U_i)$, π, U_i

$$f_{i,j} \in GL(V) \times U_{i,j} \ , \ f_{i,j} = f_{i|V\times U_{i,j}} \circ f^{-1}_{j|V\times U_{i,j}}$$

$$f_{i,j|GL(V)\times U_{i,j,k}} \circ f_{j,k|GL(V)\times U_{i,j,k}} \circ f_{k,i|GL(V)\times U_{i,j,k}} = Id_{GL(V)} \qquad \text{(cocycle condtion)}$$

In the toric world we have already given a good cover by the opens corresponing to the maxiaml cones of our fan.
$X(\Delta)$ / $M = char(T), N = char(T)^*$ / Δ^i = cones of dimension i of the fan Δ / $dim(T) = n$
$\sigma \in \Delta \to A_\sigma = k[M \cap \sigma^*], U_\sigma = Spec(A_\sigma)$

$$X(\Delta) = \bigcup_{\sigma\in\Delta^n} U_\sigma$$

So the construction of a vector bundle becomes the following problem :
$\forall \sigma, \sigma' \in \Delta^n$ such that σ and σ' share a common $(n-1)$-cone $\tau = \sigma \cap \sigma'$ find $g_{\sigma\sigma'} \in GL_s(k[A_\tau]) \subset gl_s(k[M])$ such that :

$$\forall \text{ path of closed adjacant n-cones } \sigma_0, \sigma_1, \ldots, \sigma_m, \sigma_0 \ : \ g_{\sigma_0\sigma_1} \cdots g_{\sigma_m\sigma_0} = Id$$

we have an action of $\prod_{\sigma\in\Delta^m} GL(A_\sigma)$ on the above set of bundle data by $\{g_{\sigma\sigma'}\}_{\sigma\leftrightarrow\sigma'} \xrightarrow{\{g_\sigma\}_{\sigma\in\Delta^n}} \{g_\sigma g_{\sigma\sigma'} g^{-1}_{\sigma'}\}_{\sigma\leftrightarrow\sigma'}$ Automorphisms of a bundle data correspond to stabilizer groups of the above action, and isomorphic types are given by the quotient set (the orbit space), and a non-trivial bundles are those that are not in the orbit of the trivial bundle given by $\{Id\}_{\sigma\leftrightarrow\sigma'}$.

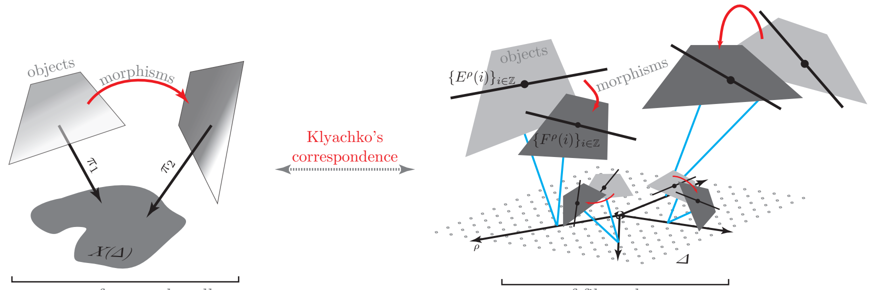
$$H^1(X(\Delta), GL_n) \ = \ \text{set of bundle data} \ / \ \textstyle\prod_{\sigma\in\Delta^m} GL(A_\sigma)$$
$$\mathfrak{H}^1(X(\Delta), GL_n) \ = \ \text{stack of principal } GL_n\text{-bundles over } X(\Delta)$$

here one needs to keep the isomorphism map and look at the functor in groupoids (no just sets)

(?) use trace formula for the action of the Frobenuis map on $X(\Delta)$ over a finite field + do some counting and mix this information with information one gets over algebraically closed fields + ...

Equivariant Toric Bundles

To approach the problem we are going to use Klyachko's equivalence between the category of vector bundles over a toric variety and the category of collection of filtered vector subspaces of a given vector space which also satisfy some compatability conditions.



category of vector bundles

category of filtered vector spaces

non-trivial equivariant vector bundles over toric variaties correspond to a collection of filteration such that one cannot find a **global splitting** for it

$E \to X(\Delta)$ $\Leftrightarrow$ $\{E^\rho(i)\}_{\rho\in\Delta^1, i\in\mathbb{Z}} = \{\{E^\rho(i)\}_{i\in\mathbb{Z}}\}_{\rho\in\Delta^1}$
where $E^\rho(i)$ are vector subspaces of $E_{x_{generic}}$ such that $E^\rho(i) \subset E^\rho(i+1)$ and $E^\rho(i) = 0$ for $i \ \rangle\rangle\rangle \ 0$ and $E^\rho(i) = E$ for $i \ \langle\langle\langle \ 0$ + they satisfy the following condition

(splitting condition)
$$\forall \sigma \in \Delta \quad \exists\, \mathbf{u} = \{u_j\}_{j\in J} \subset M(\sigma), E^\sigma(u_j) \subset E \quad s.t.$$
$$\forall \rho \in \sigma^1 \quad E^\rho(i) = \bigoplus_{\substack{u\in\mathbf{u} \\ \langle\rho,u\rangle \ge i}} E^\sigma(u)$$

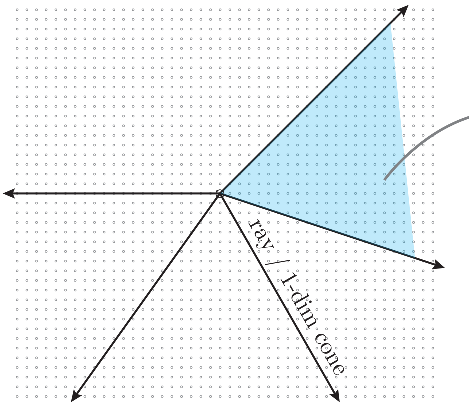
$M(\sigma) = M/\sigma^\perp$
$M = \mathrm{char}(T)$
σ^1 = set of all rays in σ

equivariant-vector bundle data

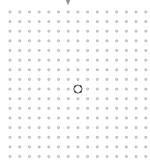
Exercise. From the equivariant-vector bundle data forget about the indices and only consider the partial flags for each ray plus the condition that partial flags of the rays of a cone must be coordinate subvector spaces for a choice of a basis. Then for any non-affine fan find a non-trivial set of data.
Exercise. From the equivariant-vector bundle data forget about the indices but keep their indices as well as their dimesnsion plus the splitting condition over all cones become a condition between the corresponding dimensions. Then for any non-affine fan find a non-trivial set of data. (multi-linear piece-wise linear function on fans)
Exercise. For any non-affine fan find a non-trivial set of data.

Properties of Toric Varieties

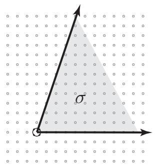
toric variety X $\longleftrightarrow$ fan Δ in a dual character lattice of a torus $char(T)^\vee$ =



$T = (\mathbb{G}_m)^n$ $\updownarrow$



affine toric $X = Spec(\Bbbk[M\sigma])$ $\updownarrow$ fan Δ = one maximal cone σ =



$X(\Delta) \times X(\Delta') = X(\Delta \times \Delta')$

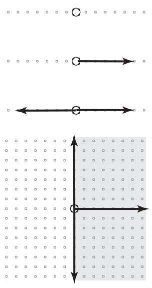
fan of the product of two toric varieties is the product of corresponding fans.

$\mathbb{G}_m$ $\longleftrightarrow$

$\mathbb{A}^1$ $\longleftrightarrow$

$\mathbb{P}^1$ $\longleftrightarrow$

$\mathbb{A}^1 \times \mathbb{P}^1$ $\longleftrightarrow$

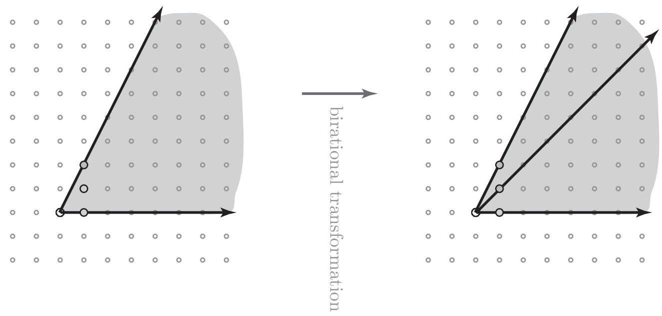


$\mathbb{P}^n(d_0, \ldots, d_n)$ $\longleftrightarrow$?

weighted projective space — exercise

smooth toric variety $\longleftrightarrow$ cones of the fan Δ can be generated by a sub-basis of the lattice $char(T)^\vee$

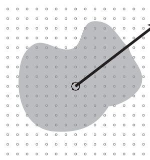
resolution of singularities of a toric variety $X(\Delta)$ $\longleftrightarrow$ subdivisions of the fan Δ



$\pi_1(X(\Delta)) = 0$ $\longleftarrow$ fan Δ has a $dim(char(T))$-dimentional cone

simply-connected

T-Weil divisors of $X(\Delta)$ $\longleftrightarrow$ $\mathbb{Z}$[rays of the fan Δ]



poset of the stratification of $X(\Delta)$ by T-orbits $\longleftrightarrow$ poset of the cones of the fan Δ

calculation of the motivic class of the $X(\Delta)$ in $K_0(Var_{\Bbbk})$

$$[X(\Delta)] = \sum_{\sigma \in \Delta} (\mathbb{L} - 1)^{\mathrm{codim}_\Delta(\sigma)}$$

$$\mathbb{L} = [\mathbb{A}^1]$$

can be used for counting calculation over tower of finite fields

Klyachko's Correspondence

$\{\{E^\rho(i)\}_{i\in\mathbb{Z}}\}_{\rho\in\Delta^1}$

$E \oplus F \to X \Leftrightarrow \{\{E^\rho(i) \oplus F^\rho(i)\}_{i\in\mathbb{Z}}\}_{\rho\in\Delta^1}$

$\{\{F^\rho(i)\}_{i\in\mathbb{Z}}\}_{\rho\in\Delta^1}$

$$\begin{array}{ccc} E & \to & F \\ & \searrow \ \swarrow & \\ & X & \end{array} \Leftrightarrow \{\{E^\rho(i)\}_{i\in\mathbb{Z}}\}_{\rho\in\Delta^1} \to \{\{F^\rho(i)\}_{i\in\mathbb{Z}}\}_{\rho\in\Delta^1}$$

$$\begin{array}{ccc} E^\rho(i) & \subset & E^\rho(i+1) \\ \downarrow & & \downarrow \\ F^\rho(i) & \subset & F^\rho(i+1) \end{array}$$