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A-posteriori Error Estimation for Finite Element Methods

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This summer I had the pleasure of working under the supervision of Dr. Dominik Schötzau in the area of finite element methods. Specifically, I studied the finite element scheme presented in [2] for a family of partial differential equations called convection-diffusion-reaction (CDR) equations with Dirichlet boundary conditions. This method is of a type called a mixed finite element method, because a finite element approximation is found for both the solution to the PDE as well as for the flux of the solution. These approximations are piecewise constant and piecewise linear, respectively. In [2], the author also introduces a postprocessed variable, which is a piecewise quadratic approximation to the solution that may be constructed solely from the data of the two finite element solutions. He also states an a-posteriori error estimator, meaning an estimator that may be computed from the data of the problem as well as the finite element approximations, in terms of this postprocessed variable.

My first goal was to examine some of the numerical and analytic properties of this finite element scheme in one dimension, where the CDR equation becomes an ordinary differential equation. I was able to prove that in one dimension the postprocessed variable is continuous, and that it also has zero boundary conditions. Continuity does not hold in general in higher dimensions, although a weaker version of continuity does. I was also able to provide a proof of an alternate form for the error estimator in the 'pure diffusion' case, when the convection and reaction terms of the CDR equation are zero. Much of my summer was spent implementing the scheme in Matlab, and compiling results from various test cases, including some with boundary layers or singularities. In particular, I was concerned with an adaptively refined mesh computed by refining elements on which the error estimator is greater than a certain percentage of the maximum elemental error estimator in each iteration. This adaptively refined mesh showed great refinement near any singularities or boundary layers and consequently performed much better in those cases than did uniform refinement. My numerical tests also confirmed that the error estimator presented in [2] is a guaranteed upper bound on the actual error, and that the various errors in the finite element solutions to the exact solution and the flux exhibit the correct orders of convergence. I was also interested in the efficiency of the estimator, defined as how much the error estimator overestimates the true error. However, in general we noted that our observed efficiencies were worse than those presented in two-dimensional test cases in [2]. This anomaly may actually be explained as a byproduct of the form of the error estimators - in general, the one-dimensional case produces worse efficiencies than the two-dimensional case for this scheme.

I also studied a primal piecewise quadratic finite element method (meaning that only one piecewise quadratic finite element solution is computed) for the one-dimensional Poisson problem with Dirichlet boundary conditions. This problem is similar to the one-dimensional pure diffusion case of the CDR equation. The basis for this study was a set of course notes created by my supervisor ([1]). This was done in hopes of comparing the performance of this error estimator with that from the mixed error estimator that I had studied previously. The two schemes have identical error measures, but the general form of the error estimator for the Poisson problem contains an undetermined constant C. In order to compare the two estimators, it was necessary that I obtain either an exact value or an upper bound on C. Dr. Schötzau and I managed to prove that for our special case of the Poisson problem, the constant C is exactly 1. This allowed me to fairly compare the two estimators in a series of numerical test cases, for which the results showed that the mixed estimator produced much better efficiencies in both uniform and adaptively refined meshes. These results led me to conclude that the error estimator of [2] is better than the primal error estimator.

References

[1] D. Schötzau, *MATH 521 Course Notes*, University of British Columbia (2009).

[2] M. Vohralík, *A posteriori error estimates for lowest-order mixed finite element discretizations of convection-diffusion-reaction equations*, SIAM J. Numer. Anal., 45 (2007), pp. 1570 - 1599.